Higgs bundles and Hermitian symmetric spaces

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A mis padres y a mi hermano

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List of Symbols

$\mathfrak{a}$	Maximal subalgebra contained in $\mathfrak{m}$, hence abelian	16
$\mathfrak{a}^{\pm}$	Image of $\mathfrak{a}$ by $\varphi^{\pm}$, subalgebra of $\mathfrak{m}^{\pm}$	18
a	Multiplicity of the roots $\pm\frac{1}{2}\gamma_j \pm \frac{1}{2}\gamma_k$ $(j \neq k)$ in Σ	25
B	Killing form	12
B_θ	Positive definite quadratic form	12
B_τ	Hermitian form defined y $B_\tau(X,Y) = B(X, \tau Y)$	22
b	Multiplicity of the roots $\pm\frac{1}{2}\gamma_j$ in Σ	25
β	Component of the Higgs field	60
C_0	Subset of roots $\{\alpha \in \Delta_C^+ \mid \pi(\alpha) = 0\}$	24
C_i	Subset of roots $\{\alpha \in \Delta_C^+ \mid \pi(\alpha) = -\frac{1}{2}\gamma_i\}$	24
C_{ij}	Subset of roots $\{\alpha \in \Delta_C^+ \mid \pi(\alpha) = \frac{1}{2}(\gamma_j - \gamma_i)\}$	24
C_{**}	Subset of compact restricted roots	44
C_G	Subgroup of $H^{\mathbb{C}}$ defined by $C_{H^{\mathbb{C}}}(\mathfrak{g}_T^{\mathbb{C}})$	52
C'_G	Semisimple part of C_G	52
C	Centralizer C of $H_T^{\mathbb{C}}$ in $H^{\mathbb{C}}$	51
c	Cayley transform, $c = exp\left(\frac{\pi}{4} i y_\Gamma\right) \in U \subset G^C$	19
χ_T	Toledo character in $\mathfrak{h}^{\mathbb{C}}$, given by $\frac{2}{N}\sum_{\alpha \in \Delta_Q^+} \alpha$	31
$\chi_{\Gamma'}$	Character defined by $\sum_{\gamma \in \Gamma'} \gamma$	45
χ'	Character defined by $\chi_T - \chi_{\Gamma'}$	45
χ^*	Character of H_T^* describing semi-invariance of det	36
${}^c\mathcal{D}$	Cayley transform of the bounded domain $\mathcal{D} \subset \mathfrak{m}^+$	23
$\mathcal{D}$	Realization of M as a bounded domain in $\mathfrak{m}^+$	15
d	Toledo invariant	61
Δ_C	Set of compact roots	17
Δ_Q	Set of non-compact roots	17
$\Delta_Q^{\pm}$	Set of positive (negative) non-compact roots	17
$\Delta_C^{\pm}$	Set of positive (negative) compact roots	17
det	Determinant of the Jordan algebra	36

Introducción: resumen y conclusiones

Sea G un grupo de Lie semisimple, y sea $\pi_1 X$ el grupo fundamental de una superficie compacta X de género g, finitamente presentado como

$$\pi_1 X = \langle a_1, b_1, \ldots, a_g, b_g \mid \prod_{j=1}^{g} [a_j, b_j] = 1 \rangle.$$

Cualquier representación $\rho : \pi_1 X \to G$ se identifica con un elemento de G^{2g} cuyas $2g$ componentes satisfacen la relación de conmutación. Por tanto, el espacio $\mathrm{Hom}(\pi_1 X, G)$ es una subvariedad de G^{2g}.

Consideremos el subconjunto $\mathrm{Hom}^+(\pi_1 X, G)$ de representaciones reductivas, es decir, aquellas cuya composición con la representación adjunta es completamente reducible, o equivalentemente, cuando G es algebraico, aquellas para las que la clausura de Zariski de la imagen es un grupo reductivo. El espacio de móduli de representaciones, o variedad de caractéres, se define como

$$\mathcal{R}(\pi_1 X, G) = \mathrm{Hom}^+(\pi_1 X, G)/G.$$

Este espacio es en general una variedad analítica real, que es algebraica cuando G es algebraico.

A cualquier representación reductiva $\rho : \pi_1 X \to G$ se puede asociar un objeto holomorfo, llamado G-fibrado de Higgs, definido de la siguiente manera. Sea H un subgrupo compacto maximal de G, y sea $\mathfrak{g} = \mathfrak{h} + \mathfrak{m}$ la descomposición de Cartan de $\mathfrak{g}$, donde el espacio vectorial $\mathfrak{m}$ es isomorfo al espacio tangente de G/H. Un G-fibrado de Higgs es un $H^{\mathbb{C}}$-fibrado principal E junto con una sección holomorfa $\varphi \in H^0(E(\mathfrak{m}^{\mathbb{C}}) \otimes K)$, donde K denota el fibrado de linea canónico. Se define una noción de poliestabilidad para G-fibrados de Higgs (Sección 3.1) que corresponde a la reductividad de las representaciones. Decimos que dos G-fibrados de Higgs (E, φ) y (E', φ') son isomorfos si existe un isomorfismo $f : E \to E'$ tal que $\varphi' = f^* \varphi$, donde f^* es la aplicación $E(\mathfrak{m}^{\mathbb{C}}) \otimes K \to E'(\mathfrak{m}^{\mathbb{C}}) \otimes K$ inducida por f. El espacio de móduli de G-fibrados de Higgs poliestables $\mathcal{M}(G)$ es por definición el conjunto de clases de isomorfía de G-fibrados de Higgs poliestables (E, φ). Este espacio tiene estructura de variedad analtica compleja, que es algebraica cuando G es algebraico.

Los espacios de móduli $\mathcal{R}(\pi_1 X, G)$ y $\mathcal{M}(G)$ son homeomorfos. Este resultado es una consecuencia de una correspondencia entre cuatro espacios. El espacio de móduli $\mathcal{R}(\pi_1 X, G)$ es homeomorfo al espacio de móduli de G-conexiones planas reductivas, donde se pueden resolver las ecuaciones de armonicidad, como fue probado por Donaldson para $\mathrm{SL}(2, \mathbb{C})$ ([Don87]), y generalizado por Corlette ([Cor88]) y Labourie ([Lab91]). Por otra parte, el espacio de móduli $\mathcal{M}(G)$ es homeomorfo al espacio de móduli de soluciones a las ecuaciones de Hitchin, como probaron Hitchin para $\mathrm{SL}(2, \mathbb{C})$ ([Hit87]), Simpson para un grupo de Lie complejo reductivo arbitrario ([Sim92]), y Bradlow, García-Prada, Gothen y Mundet i Riera para un grupo de Lie real reductivo ([BGM03], [GGM09]). Finalmente, hay una correspondencia entre soluciones a las ecuaciones de Hitchin y de armonicidad. Cuando G es compacto, el campo de Higgs es nulo y esta correspondencia fue establecida por Narasimhan y Seshadri para $\mathrm{SU}(n)$ ([NS65]), y Ramanathan para un grupo de Lie compacto semisimple ([Ram75]). Estas correspondencias entrelazan las estructuras topológicas, geométrico-diferenciales y geométrico-algebráicas de X por medio de teoremas de existencia de soluciones para ecuaciones gauge no lineales en derivadas parciales. En la Sección 5.1 se indica una prueba de esta correspondencia.

Si G es un grupo conexo, se puede asociar un invariante topológico en $\pi_1 G$ a cualquier representación $\rho : \pi_1 X \to G$ de la siguiente manera. Por una parte, a partir del $\pi_1 X$-fibrado principal $\widetilde{X} \to X$, consideramos el G-fibrado principal plano asociado $E_\rho = \widetilde{X} \times_\rho G$. Por otra parte, de la sucesión exacta

$$1 \to \pi_1 G \to \widetilde{G} \to G \to 1$$

obtenemos la suceción exacta larga en cohomología, y, en particular, el morfismo de conexión

$$H^1(X, G) \xrightarrow{c} H^2(X, \pi_1 G),$$

donde G y $\pi_1 G$ denotan los haces de funciones localmente constantes en G y $\pi_1 G$ respectivamente, el dominio parametriza clases de equivalencia de G-fibrados principales planos sobre X, y el codominio es isomorfo a $\pi_1 G$ por el teorema de los coeficientes universales puesto que $\dim_{\mathbb{R}} X = 2$ y el grupo fundamental de un grupo de Lie es Abeliano. Además, $\pi_1 G \cong \pi_1 H$ puesto que H es un retracto de deformación de G. La clase asociada a ρ es $c(E_\rho) \in \pi_1 H$. Este invariante separa el espacio de móduli, es decir, elementos con valores distintos del invariante están en componentes conexas distintas.

Una clase de grupos de interés especial es el de los grupos de Lie semisimples no compactos de tipo Hermítico con centro finito. Estos son grupos de Lie G con centro finito tales que el cociente G/H por un subgrupo compacto maximal H es un espacio simétrico Hermítico de tipo no compacto. Esta clase está formada por los grupos con centro finito asociados a las álgebras de Lie $\mathfrak{su}(p,q)$, $\mathfrak{sp}(2n,\mathbb{R})$, $\mathfrak{so}^*(2n)$, $\mathfrak{so}(2,n)$, $\mathfrak{e}_6^{-14}$ y $\mathfrak{e}_7^{-25}$. En las álgebras de Lie excepcionales, el superíndice se refiere a la signatura de la forma de Killing (ver, p.ej., [Hel01]). Cuando G es Hermítico, la estructura casi-compleja inducida separa $\mathfrak{m}^{\mathbb{C}}$ en autoespacios de valores propios $(\pm i)$, $\mathfrak{m}^+$ y $\mathfrak{m}^-$. Un G-fibrado de Higgs tiene entonces dos componentes $\beta \in H^0(E(\mathfrak{m}^+)\otimes K)$ y $\gamma \in H^0(E(\mathfrak{m}^-) \otimes K)$. Como el centro de H es un círculo, $\pi_1 H$ es isomorfo a $\mathbb{Z}$ junto con una posible parte de torsión (que solo aparece en el caso de $\mathfrak{so}(2,n)$). La proyección a $\mathbb{Z}$ define una invariante que probamos que está acotado. Los fibrados cuyo invariante alcanza el valor maximal (positivo o negativo) son llamados G-fibrados de Higgs maximales. Como en el caso de representaciones, este invariante separa $\mathcal{M}(G)$ en subvariedades. El objetivo principal de esta tesis es el estudio del espacio de móduli $\mathcal{M}_{max}(G)$ de G-fibrados de Higgs poliestables con valor maximal de este invariante, para G un grupo de Lie no compacto de tipo Hermítico con centro finito.

Aunque no en el contexto de espacios de móduli, este invariante fue estudiado en [Mil58], donde Milnor relacionó la clase de Euler de un $\mathrm{GL}^+(2,\mathbb{R})$-fibrado sobre X con la existencia de una conexión plana en X. Milnor probó que tal conexión existe si y solo si la clase de Euler e satisface $|e| \leq g-1$. De hecho, esta clase de Euler coincide con el invariante definido anterioremente proyectando $\pi_1 G$ a $\mathbb{Z}$. Este resultado fue generalizado por Wood en [Woo71] para grupos de homeomorfismos y difeomorfismos del círculo, lo cual justifica el nombre dado a la desigualdad. En [Gol80], [Gol82] y [Gol88], Goldman estudió el espacio de representaciones $\mathrm{Hom}(\pi_1 X, \mathrm{PSL}(2,\mathbb{R}))$ y contó el número de componentes conexas. Goldman utilizó el número de Euler como invariante y probó que representaciones con invariante maximal son discretas y fieles. Identificó además la subvariedad de representaciones maximales con el espacio de Teichmüller. Estas componentes ya habían atraído la atención de Weil en su estudio del espacio de representaciones de un grupo abstracto en un grupo topológico. ([Wei60], [Wei62]).

Nuestro enfoque usando fibrados de Higgs sigue el trabajo de Hitchin ([Hit87]) para $\mathrm{SL}(2,\mathbb{R}) \cong \mathrm{SU}(1,1)$. Un $\mathrm{SL}(2,\mathbb{R})$-fibrado de Higgs es equivalente a un fibrado vectorial holomorfo $V = L \oplus L^{-1}$, donde L es un fibrado de línea, junto con un campo de Higgs $\varphi \in H^0(X, \operatorname{End}_0 V \otimes K)$ con dos componentes $\beta : L \to L^{-1} \otimes K$ y $\gamma : L^{-1} \to L \otimes K$, donde K denota el fibrado de línea canónica sobre X.

$$\varphi = \begin{pmatrix} 0 & \gamma \\ \beta & 0 \end{pmatrix} \in H^0(X, \operatorname{End}_0 V \otimes K).$$

En este caso, la cota para el invariante es una consecuencia de la condición de semiestabilidad del fibrado de Higgs. Hitchin también estudió los objetos con invariante maximal. Probó que para un fibrado de Higgs estable maximal (positivo), L debe ser una raíz cuadrada de K, el campo β debe ser la identidad y el campo γ es una diferencial cuadrática, es decir, $\gamma \in H^0(X, K^2)$. Además, identificó la componente maximal con el espacio de Teichmüller. Hitchin definió posteriormente ([Hit92]) las componentes de Hitchin-Teichmüller para formas reales split de grupos de Lie complejos simples, lo que incluye el grupo $\mathrm{Sp}(2n, \mathbb{R})$, tanto split como Hermítico, y para el cual las componentes de Hitchin-Teichmüller son maximales.

El hecho de que para fibrados de Higgs maximales tengamos $L = K^{1/2}$, $\beta = \mathrm{id}$ y $\gamma \in H^0(X, K^2)$, y las condiciones sobre representaciones maximales (las cuales son discretas y fieles) reveló un fenómeno de rigidez para objetos maximales que ha sido ampliamente estudiado. En vez del invariante entero definido anteriormente como la proyección de la clase topológica a $\mathbb{Z}$, utilizaremos el invariante de Toledo. Como se muestra en la Sección 3.5, el primero es simplemente un múltiplo entero del invariante de Toledo, que puede ser racional. Toledo definió en [Tol89] un invariante para una representación $\rho : \pi_1 X \to \mathrm{PU}(1, n)$ de la siguiente manera. El grupo $\mathrm{PU}(1, n)$ es el grupo de isometrías del espacio hiperbólico $\mathbf{H}^n_{\mathbb{C}}$, espacio que se identifica con el espacio simétrico $\mathrm{PU}(1, n)/\mathrm{P}(U(1) \times U(n))$. Sea ω la forma de Kähler correspondiente a una métrica Hermítica (normalizada) de curvatura seccional holomorfa minimal -1, la métrica de Bergmann.

La representación $\rho : \pi_1 X \to \mathrm{PU}(1, n)$ determina un *developing map* ρ-equivariante $f : \widetilde{X} \to \mathbf{H}^n_{\mathbb{C}}$. El *pullback* $f^*\omega$ define una forma $\pi_1 X$-invariante en $\widetilde{X}$ y por tanto desciende a una forma en X que también denotamos por $f^*\omega$. El invariante de Toledo se define como

$$T(\rho) = \int_X f^*\omega.$$

Este proceso puede generalizarse reemplazando el espacio hiperbólico por un espacio simétrico Hermítico no compacto G/H con su métrica de Bergmann. Domic y Toledo ([DT87]) ya habían probado, encontrando una cota a la norma de la clase de Kähler class del espacio simétrico, que el invariante de Toledo satisface una desigualdad de tipo Milnor-Wood. Cuando este invariante alcanza sus valores maximales, Toledo detectó cierta rigidez consistente en la estabilización de una geodésica compleja, y por tanto la imagen de una representación maximal está dentro del subgrupo $\mathrm{P}(\mathrm{U}(1,1)\times \mathrm{U}(n-1))$. La geodésica compleja corresponde a subespacio de tipo tubo maximal de $\mathbf{H}^n_{\mathbb{C}}$. Este trabajo fue generalizado por Hernández en [Her91] para el grupo $\mathrm{PSU}(2,q)$. La generalización para $\mathrm{PU}(p,q)$ fue obtenida usando técnicas de fibrados de Higgs por Bradlow, García-Prada y Gothen en [BGG01]. Este trabajo fue seguido por el estudio de $\mathrm{U}(p,q)$, que incluye el caso de $\mathrm{SU}(p,q)$, en [BGG03] y $\mathrm{SO}^*(4n+2)$ ([BGG06], [BGG12]). Un enfoque general distinto usando cohomología acotada ha sido usado por Burger, Iozzi, Wienhard, como se explica en [BIW10b]. Este enfoque permite definir también el invariante de Toledo en superficies con frontera.

Como hemos dicho anteriormente, en esta tesis estudiamos el espacio de móduli de G-fibrados de Higgs poliestables sobre una superficie de Riemann X, cuando G es un grupo de Lie real no compacto simple de tipo Hermítico con centro finito. Nuestros principales resultados establecen una cota del invariante de Toledo y describen fenómenos de rigidez inherentes a G-fibrados de Higgs maximales, aquellos cuyo invariante de Toledo alcanza una de las cotas. Generalizamos el trabajo descrito anteriormente para grupos arbitrarios de tipo Hermítico y damos una prueba independiente de la clasificación basada en cómo la geometría del espacio simétrico Hermítico asociado a G se refleja en el espacio de móduli. Para hacer esto, adoptamos un nuevo enfoque para el invariante de Toledo, definiendo un carácter que llamamos el carácter de Toledo.

En el Capítulo 2, presentamos algunos resultados básicos sobre espacios simétricos Hermíticos, siguiendo [Hel01], [FKK$^+$00] y [Ji05]. Como nuestro objetivo es el estudio de G-fibrados de Higgs, nos centramos en el grupo G además de en el espacio simétrico G/H. Los recubridores finitos o cocientes de un grupo Hermítico dan el mismo espacio simétrico Hermítico, pero los G-fibrados de Higgs asociados son objetos distintos. El *embedding* de Harish-Chandra da una realización de G/H como dominio simétrico acotado en $\mathfrak{m}^+$ y la transformada de Cayley generalizada da una realización alternativa como semiplano generalizado ([KW65]). Este semiplano puede ser biholomorfo a un dominio de tipo tubo sobre un cono Ω, en cuyo caso decimos que G es de tipo de tubo. En caso contrario, decimos que G es de tipo no tubo. Por la clasificación de espacios simétricos irreducibles de Cartan([Car26], [Car27]), los grupos correspondientes a $\mathfrak{sp}(2n,\mathbb{R})$, $\mathfrak{so}^*(4m)$, $\mathfrak{so}(2,n)$, $\mathfrak{su}(p,p)$, y $\mathfrak{e}_7^{-25}$ son de tipo tubo, mientras que $\mathfrak{so}^*(4m+2)$, $\mathfrak{su}(p,q)$ para $p\neq q$ y $\mathfrak{e}_6^{-14}$ son de tipo no tubo. Esta realización da los ingredientes con los que se enuncian los resultados principales. Por una parte, para grupos de tipo tubo, $\mathfrak{m}^+$ tiene estructura de álgebra de Jordan unitaria (con unidad e_Γ, ver Sección

2.4.2) y determinante det, y el cono Ω es el espacio simétrico H^*/H'_0, donde H^* es el dual no compacto de H y H'_0 es la componente conexa de la identidad del estabilizador de e_Γ en H^*. Por otra parte, cada grupo de tipo no tubo tiene un subgrupo maximal de tipo tubo. En particular, esto permite definir una noción de rango en $\mathfrak{m}^+$, aun cuando el determinante no existe. Análogamente se puede hacer para $\mathfrak{m}^-$.

La condición de tipo tubo también se refleja en el sistema de raíces de $\mathfrak{g}$. Como G es Hermítico, se puede escoger una subalgebra de Cartan $\mathfrak{t}^{\mathbb{C}}$ contenida en $\mathfrak{h}^{\mathbb{C}}$, de forma que cada raíz $\alpha \in \Delta(\mathfrak{g}^{\mathbb{C}}, \mathfrak{t}^{\mathbb{C}})$ satisface que su espacio de raíces $\mathfrak{g}^{\mathbb{C}}_\alpha$ está o bien en $\mathfrak{h}^{\mathbb{C}}$, y decimos que la raíz es compacta (y escribimos $\alpha \in \Delta_C$), o bien en $\mathfrak{m}^{\mathbb{C}}$, y decimos que la raíz es no compacta (y escribimos $\alpha \in \Delta_Q$). Las raíces de $\Delta(\mathfrak{g}^{\mathbb{C}}, \mathfrak{t}^{\mathbb{C}})$ se restringen a una cierta subalgebra $i\mathfrak{t}^- \subset \mathfrak{t}^{\mathbb{C}}$ dada por una elección de raíces no compactas. Los elementos resultantes forman un sistema de raíces no necesariamente reducido, llamado sistema de raíces restringidas, que se estudia en la Sección 2.3. Este sistema solo puede ser de dos tipos: reducido, que corresponden a grupos de tipo tubo, y no reducido, que corresonde a grupos de tipo no tubo. Algunas raíces pueden proyectar a la misma raíz restringida, y las multiplicidades posibles de una raíz restringida son invariantes del sistema. En términos de estas multiplicidades, el número dual de Coxeter N, un invariante asociado al álgebra de Lie $\mathfrak{g}$, se define en la Sección 2.3.1.

Un sistema de raíces de un grupo de tipo no tubo G siempre contiene un sistema de tipo tubo maximal que da la subalgebra $\mathfrak{g}^{\mathbb{C}}_T \subset \mathfrak{g}^{\mathbb{C}}$, la subalgebra $\mathfrak{h}^{\mathbb{C}}_T \subset \mathfrak{h}^{\mathbb{C}}$ con su subgrupo correspondiente $H^{\mathbb{C}}_T \subset H^{\mathbb{C}}$, y el espacio vectorial $\mathfrak{m}_T$.

El carácter de Toledo χ_T se define en la Sección 2.4. Se define como un carácter en el álgebra de Lie $\mathfrak{h}^{\mathbb{C}}$ usando el sistema de raíces de $\mathfrak{g}^{\mathbb{C}}$. La suma de las raíces no compactas positivas es un elemento del dual de $\mathfrak{t}^{\mathbb{C}}$, y consideramos el múltiplo racional

$$\chi_T = \frac{2}{N} \sum_{\alpha \in \Delta_Q^+} \alpha.$$

Equivalentemente, el dual s_{χ_T} de χ_T con respecto a la forma de Killing viene dado por un elemento de $\mathfrak{t}^{\mathbb{C}}$. El Lemma 2.37 muestra que este elemento está de hecho en el centro, $s_{\chi_T} \in i\mathfrak{z} \subset \mathfrak{z}^{\mathbb{C}} \cong \mathfrak{h}^{\mathbb{C}}/[\mathfrak{h}^{\mathbb{C}}, \mathfrak{h}^{\mathbb{C}}]$, y por tanto define un carácter de $\mathfrak{h}^{\mathbb{C}}$. La exponenciación de este carácter al grupo $H^{\mathbb{C}}$ depende de cuatro invariantes. Dos que vienen del álgebra de Lie $\mathfrak{g}$: la dimensión de $\mathfrak{m}$ y el número dual de Coxeter. Y dos que vienen del grupo G: el número finito $l = |Z_0^{\mathbb{C}} \cap [H^{\mathbb{C}}, H^{\mathbb{C}}]|$, donde Z_0 es la componente conexa de la identidad del centro de $H^{\mathbb{C}}$, y el orden de la exponenciación del elemento $2\pi J$ como elemento de G, donde $J \in \mathfrak{z}(\mathfrak{h})$ es el elemento que da la estructura casi compleja en $\mathfrak{m}$ vía la acción adjunta. La Proposición 2.39 da la condición numérica para que el carácter $q \cdot \chi_T$ exponencie a $H^{\mathbb{C}}$:

$$\frac{q \cdot \dim \mathfrak{m} \cdot o(e^{2\pi J})}{l \cdot N} \in \mathbb{Z}.$$

El elemento unidad del álgebra de Jordan, $e_\Gamma \in \mathfrak{m}^+$, define una subálgebra parabólica de $\mathfrak{h}^{\mathbb{C}}$ dada por

$$\mathfrak{p}_{e_\Gamma} = \operatorname{Ker}(\operatorname{ad}(e_\Gamma)_{|\mathfrak{h}^{\mathbb{C}}}) \oplus \operatorname{Im}(\operatorname{ad}(e_\Gamma)_{|\mathfrak{m}^-}),$$

para la cual el carácter de Toledo es un carácter antidominante, como muestra la Proposición 2.60. Este resultado se extiende por conjugación a cualquier elemento de determinante no nulo en $\mathfrak{m}^+$. La prueba de estos resultados requiere algunos resultados técnicos sobre sistemas de raíces restringidas que se prueban en la Sección 2.3.

Para grupos de tipo tubo, se tiene que un múltiplo racional del carácter de Toledo $q_T \cdot \chi_T$ levanta a un carácter del grupo, $\widetilde{\chi}_T$, y este nuevo carácter describe la semi-invariancia del determinante de $\mathfrak{m}^+$ con respecto a la acción de $H^{\mathbb{C}}$,

$$\det(h \cdot x)^{q_T} = \widetilde{\chi}_T(h) \det(x)^{q_T},$$

para $h \in H^{\mathbb{C}}$ y $x \in \mathfrak{m}^+$. Así, la acción de $H^{\mathbb{C}}$ preserva los elementos de determinante no nulo, $\mathfrak{m}^+_{D\neq 0}$. Además, $H^{\mathbb{C}}$ actúa transitivamente en estos elementos y $\mathfrak{m}^+_{D\neq 0}$ es el espacio homogéneo $H^{\mathbb{C}}/H'^{\mathbb{C}}$, donde H' es el estabilizador en H del elemento e_Γ.

El estudio de G-fibrados de Higgs (E, β, γ) α-poliestables para G de tipo Hermítico y α un parámetro en el centro de $\mathfrak{h}^{\mathbb{C}}$ empieza en el Capítulo 3. La noción de α-poliestabilidad se da en la Definición 3.5 en términos de reducciones del grupo de estructura de E, $H^{\mathbb{C}}$, a subgrupos parabólicos de Richardson (estudiados en la Sección 2.5) y sus carácteres antidominantes. El invariante de Toledo se define a partir del carácter de Toledo como

$$d = \frac{1}{q_T} \deg(E(\widetilde{\chi}_T)).$$

Esta definición coincide con las definiciones del invariante de Toledo dadas anteriormente, como se muestra en la Sección 5.2. Además, de la noción de rango del álgebra de Jordan $\mathfrak{m}^+$, y también $\mathfrak{m}^-$, definimos un rango para las componentes del campo de Higgs $\beta \in H^0(E(\mathfrak{m}^+) \otimes K)$ y $\gamma \in H^0(E(\mathfrak{m}^-) \otimes K)$.

La desigualdad de tipo Milnor-Wood que se prueba en la Sección 3.4 da una cota del invariante de Toledo d de un G-fibrado de Higgs semiestable (E, β, γ) dependiente del parámetro y de los rangos de las dos componentes del campo. Tenemos el siguiente teorema.

Theorem (3.18). *Sea $\alpha \in i\mathfrak{z}$ tal que $\alpha = i\lambda J$ para $\lambda \in \mathbb{R}$. Sea (E, β, γ) un G-fibrado de Higgs α-semistable. Entonces, el invariante de Toledo $d = \frac{1}{q_T}\deg(E(\widetilde{\chi}_T))$ satisface:*

$$-\operatorname{rk}(\beta)(2g-2) - \left(\frac{2\dim\mathfrak{m}}{N} - \operatorname{rk}(\beta)\right)\lambda \leq d \leq \operatorname{rk}(\gamma)(2g-2) + \left(\frac{2\dim\mathfrak{m}}{N} - \operatorname{rk}(\gamma)\right)\lambda,$$

donde N es el número dual de Coxeter y $\dim\mathfrak{m}$ es la dimensión de la representación de isotropía de G. En el caso de tipo tubo, esto es simplemente

$$-\operatorname{rk}(\beta)(2g-2) - (r - \operatorname{rk}(\beta))\lambda \leq d \leq \operatorname{rk}(\gamma)(2g-2) + (r - \operatorname{rk}(\gamma))\lambda.$$

La prueba que damos es independiente del teorema de clasifación y está basada en la condición de semiestabilidad y la estructura de álgebra de Jordan de un subtubo.

Como consecuencia, obtenemos la desigualdad de Milnor-Wood.

Theorem (4.1)**.** *Sea G un grupo simple de tipo Hermítico. Sea d el invariante de Toledo de un G-fibrado de Higgs semistable. Entonces,*

$$|d| \leq \operatorname{rk}(G/H)(2g-2).$$

Consideremos que el parámetro α es nulo de ahora en adelante. Este caso es de especial interés por la conexión con representaciones de grupos de superficies descrita anteriormente. Pese a ello, el estudio de la estabilidad dependente de un parámetro puede ser de ayuda cuando se utiliza teoría de Morse para contar el número de componentes conexas del espacio de móduli, como se muestra en [BGG03].

El Capítulo 4 está dedicado al estudio del espacio de móduli $\mathcal{M}_{max}(G)$ de G-fibrados de Higgs poliestables maximales, es decir, $d = \pm\operatorname{rk}(G/H)(2g-2)$, y contiene los resultados principales de la tesis, que describen la rigidez de G-fibrados de Higgs maximales.

Cuando G es de tipo tubo, $\mathcal{M}_{max}(G)$ se embebe en el epacio de móduli de H^*-pares de Higgs K^2-*twisted*, donde H^* es el dual no compacto de H que da el cono $\Omega \cong H^*/H_0'$ relacionado con G/H.

Theorem (4.7). *Sea G un grupo Hermítico de tipo tubo y H un compacto maximal. Sea H^* el dual no compacto de H en $H^{\mathbb{C}}$. Sea J el elemento del centro del álgebra de Lie $\mathfrak{g}$ que da la estructura casi compleja en $\mathfrak{m}$ (ver Proposición 2.2). Si el orden de $e^{2\pi J} \in H^{\mathbb{C}}$ divide a $(2g-2)$, entonces hay una inyección de variedades algebraicas complejas*

$$\mathcal{M}_{\max}(G) \to \mathcal{M}_{K^2}(H^*).$$

Además, G-fibrados de Higgs estables corresponden a H^*-pares de Higgs K^2-twisted.

Creemos que esta inyección es realmente un isomorfismo, como se demuestra para $\mathrm{SU}(p,p)$, $\mathrm{Sp}(2n,\mathbb{R})$ y $\mathrm{SO}^*(4m)$ en [BGG03], [GGM08] y [BGG12], respectivamente.

Cuando G es de tipo no tubo, obtenemos el siguiente teorema.

Theorem (4.15). *Sea G un grupo simple de tipo Hermítico no tubo y sea H un subgrupo compacto maximal. Entonces, no hay G-fibrados de Higgs estables con invariante de Toledo maximal. De hecho, cada G-fibrado de Higgs poliestable reduce a un $N_G(\mathfrak{g}_T)_0$-fibrado de Higgs estable, donde $N_G(\mathfrak{g}_T)_0$ es la componente conexa de la identidad del normalizador de $\mathfrak{g}_T$ en G.*

Así, el grupo de estructura de los G-fibrados de Higgs maximales reduce a un subgrupo estrictamente más pequeño que G. Por lo tanto, la dimensión del espacio de móduli es más pequeña que la dimensión esperada.

Además, la segunda parte de la Sección 4.3 está dedicada al espacio de móduli de G-fibrados de Higgs poliestables maximales y su realización como fibración de otros espacios de móduli. Los resultados técnicos necesarios se desarrollan en la Sección 2.6.

Estos fenómenos de rigidez han sido ampliamente estudiados, también desde el punto de vista de representaciones. El estudio de representaciones maximales ha atraído un gran interés por su significado geométrico. En el caso de $\mathrm{SL}(2,\mathbb{R})$, Goldman ([Gol80]) probó que hay 2^{2g} componentes maximales en el espacio de móduli de representaciones, las cuales pueden ser identificadas con el espacio de Teichmüller, y consisten en su totalidad en representaciones discretas y fieles. Usando métodos de cohomología acota, Burger, Labourie, Iozzi y Wienhard han probado en general que las componentes maximales para grupos de tipo Hermítico consisten enteramente de representaciones discretas y fieles. Un resultado interesante que surge desde este enfoque es que las representaciones maximales son necesariamente reductivas, así que la hipótesis de reductividad de fibrados de Higgs se satisface en el caso maximal.

Para grupos de tipo tubo, la correspondencia de Cayley $\mathcal{M}_{max}(G) \to \mathcal{M}_{K^2}(H^*)$ probada en el Teorema 4.7, muestra la rigidez de los objetos maximales. El H^*-par de Higgs K^2-*twisted* correspondiente tiene un grupo de estructura más pequeño que el G-fibrado de Higgs inicial. Aunque no es una reducción del grupo de estructura, la dimensión del nuevo grupo H^* es igual a la dimensión del subgrupo compacto maximal H de G. Además, esta correspondencia revela nuevos invariantes para G-fibrados de Higgs maximales y representaciones del grupo fundamental de una superficie en G. Estos nuevos invariantes vienen del grupo H^*. Por ejemplo, cuando $G = \mathrm{Sp}(2n,\mathbb{R})$, tenemos que $H^* = \mathrm{GL}(n,\mathbb{R})$ con $H' = \mathrm{O}(n)$ como subgrupo compacto maximal. Así, a un $\mathrm{Sp}(2n,\mathbb{R})$-fibrado de Higgs podemos asociarle la primera y la segunda clases de Stiefel-Whitney del $\mathrm{GL}(n,\mathbb{R})$-par de Higgs correspondiente vía la correspondencia de Cayley, $w_1 \in H^1(X,\mathbb{Z}/2)$, $w_2 \in H^2(X,\mathbb{Z}/2)$. Estos invariantes proporcionan una manera para contar las componentes conexas del espacio de móduli $\mathcal{M}_{max}(\mathrm{Sp}(2n,\mathbb{R}))$, como se recoge en [Got11]. Estos invariantes han aparecido desde el punto de vista de representaciones en el trabajo de Guichard y Wienhard, [GW09], donde se definen análogos de w_1, w_2 para representaciones $(\mathrm{Sp}(2n,\mathbb{R}),\mathrm{GL}(n,\mathbb{R}))$-Anosov, de las que las representaciones maximales son un caso particular.

Para los casos excepcionales tenemos los siguiente teoremas.

Theorem (4.13)**.** *Existe un imbedding de variedades algebraicas complejas*

$$\mathcal{M}_{max}(\mathrm{E}_7^{-25}) \to \mathcal{M}_{K^2}(\mathrm{E}_6^{-26} \ltimes \mathbb{R}^*).$$

Theorem (4.17)**.** *Todo E_6^{-14}-fibrado de Higgs maximal es estrictamente poliestable y reduce a un* $\mathrm{Spin}_0(2,8)\times \mathrm{U}(1)$*-fibrado de Higgs estable, y por tanto es el producto de un* $\mathrm{Spin}_0(2,8)$*-fibrado de Higgs y un fibrado de línea. Además, el* $\mathrm{Spin}_0(2,8)$*-fibrado de Higgs es maximal.*

El capítulo 5 trata sobre la relación de fibrados de Higgs con representaciones. Empezamos dando un esquema de la prueba de la equivalencia del espacio de móduli de G-fibrados de Higgs sobre X poliestables con el espacio de móduli de representaciones de $\pi_1 X$ en G. Entonces mostramos la equivalencia de distintas nociones del invariante de Toledo. En la Sección 5.3 usamos la correspondencia $\mathcal{R}(\pi_1 X, G) \cong \mathcal{M}(G)$ para enunciar los resultados principales en términos de representaciones, y revisamos enfoques previos y alternativos a la rigidez de representaciones maximales. Para grupos de tipo no tubo tenemos el siguiente teorema.

Theorem (5.4)**.** *Sea $\rho : \pi_1 X \to G$ una representación maximal del grupo fundamental de una superficie de Riemann X en un grupo de Lie semisimple de tipo Hermítico no tubo G. Entonces, la imagen de ρ está contenida en $N_G(\mathfrak{g}_T)_0$, donde $\mathfrak{g}_T$ es la subálgebra de $\mathfrak{g}$ correspondiente al subdominio de tipo tubo maximal G_T/H_T de G/H.*

Como consecuencia de este teorema, tenemos el siguiente.

Theorem (5.5)**.** *La imagen de cualquier representación maximal $\rho : \pi_1 X \to E_6^{-14}$ está contenida en* $\mathrm{Spin}_0(2,8)\times \mathrm{U}(1)$.

La tesis acaba con tres apéndices. En el primero se describe la descomposición de Cartan y el sistema de raíces restringidas para las álgebras de Lie de tipo Hermítico clásicas y excepcionales. El segundo apéndice contiene algunas notas adicionales, mientras que el tercero muestra varias tablas sobre los ingredientes de la correspondencia de Cayley y fibrados de Higgs.

Chapter 1

Introduction: summary and main results

Let G be a semisimple Lie group, and let $\pi_1 X$ be the fundamental group of a compact surface X of genus g, which is finitely presented as

$$\pi_1 X = \langle a_1, b_1, \dots, a_g, b_g \mid \prod_{j=1}^{g} [a_j, b_j] = 1 \rangle.$$

Every representation $\rho : \pi_1 X \to G$ is identified with an element of G^{2g} whose $2g$ components satisfy the commutation relation. Consequently, the space $\mathrm{Hom}(\pi_1 X, G)$ is a subvariety of G^{2g}. Consider the subset $\mathrm{Hom}^+(\pi_1 X, G)$ of reductive representations, i.e., those which are completely reducible when composed with the adjoint representation, or equivalently, when G is algebraic, those for which the Zariski closure of the image is a reductive group. One defines the moduli space of representations, or character variety, as

$$\mathcal{R}(\pi_1 X, G) = \mathrm{Hom}^+(\pi_1 X, G)/G.$$

This space is in general a real analytic variety, which is algebraic when G is algebraic.

To any reductive representation $\rho : \pi_1 X \to G$ one can associate a holomorphic object, called a G-Higgs bundle, which is defined as follows. Let H be a maximal compact subgroup of G and let $\mathfrak{g} = \mathfrak{h} + \mathfrak{m}$ be the Cartan decomposition of $\mathfrak{g}$, where the vector space $\mathfrak{m}$ is isomorphic to the tangent space of G/H. A G-Higgs bundle consists of a principal $H^{\mathbb{C}}$-bundle E together with a holomorphic section $\varphi \in H^0(E(\mathfrak{m}^{\mathbb{C}}) \otimes K)$, where K denotes the canonical line bundle. One defines a notion of polystability for G-Higgs bundles (Section 3.1) which corresponds to the reductivity of representations. We say that two G-Higgs bundles (E, φ) and (E', φ') are isomorphic if there is an isomorphism $f : E \to E'$ such that $\varphi' = f^*\varphi$, where f^* is the map $E(\mathfrak{m}^{\mathbb{C}}) \otimes K \to E'(\mathfrak{m}^{\mathbb{C}}) \otimes K$ induced by f. The moduli space of polystable G-Higgs bundles $\mathcal{M}(G)$

is by definition the set of isomorphism classes of polystable G-Higgs bundles (E, φ). This space has the structure of a complex analytic variety, which is algebraic when G is algebraic.

The moduli spaces $\mathcal{R}(\pi_1 X, G)$ and $\mathcal{M}(G)$ are homeomorphic. This is a consequence of a fourfold correspondence. The moduli space $\mathcal{R}(\pi_1 X, G)$ is homeomorphic to the moduli spaces of reductive flat G-connections, where one can solve harmonicity equations, as proved by Donaldson for $\mathrm{SL}(2, \mathbb{C})$ ([Don87]), and generalized by Corlette ([Cor88]) and Labourie ([Lab91]). On the other hand, the moduli space $\mathcal{M}(G)$ is homeomorphic to the moduli space of solutions to the Hitchin equations, as proved by Hitchin for $\mathrm{SL}(2, \mathbb{C})$ ([Hit87]), Simpson for an arbitrary complex reductive Lie group ([Sim92]), and by Bradlow, García-Prada, Gothen and Mundet i Riera for a real reductive Lie group ([BGM03], [GGM09]). Finally, there is a correspondence between solutions to the Hitchin and harmonicity equations. When G is compact, the Higgs field is zero and this correspondence had been established in the work of Narasimhan and Seshadri for $\mathrm{SU}(n)$ ([NS65]), and Ramanathan for a compact semisimple Lie group ([Ram75]). These correspondences intertwine the topological, differential geometric and algebraic geometric structures of X in a deep way, by means of existence theorems for gauge non-linear partial differential equations. A proof is sketched in Section 5.1.

If G is a connected group, one can associate a topological invariant in $\pi_1 G$ to any representation $\rho : \pi_1 X \to G$ as follows. On the one hand, from the principal $\pi_1 X$-bundle, $\widetilde{X} \to X$, we consider the associated flat principal G-bundle $E_\rho = \widetilde{X} \times_\rho G$. On the other hand, from the exact sequence

$$1 \to \pi_1 G \to \widetilde{G} \to G \to 1$$

we obtain the long exact sequence in cohomology, and, in particular, the connection map

$$H^1(X, G) \xrightarrow{c} H^2(X, \pi_1 G),$$

where G and $\pi_1 G$ denote the sheaves of locally constant functions in G and $\pi_1 G$ respectively, the domain parameterizes equivalence classes of flat principal G-bundles on X, and the target is isomorphic to $\pi_1 G$ by the universal coefficient theorem since $\dim_{\mathbb{R}} X = 2$ and the fact that the fundamental group of a Lie group is Abelian. Moreover, $\pi_1 G \cong \pi_1 H$ since H is a deformation retract of G. The class associated to ρ is $c(E_\rho) \in \pi_1 H$. This invariant separates the moduli space, i.e., elements with different values of the invariant must lie in different connected components.

A class of groups of particular interest is that of non-compact semisimple Lie groups of Hermitian type with finite centre. These are Lie groups G with finite centre such that the quotient G/H by a maximal compact subgroup H is a Hermitian symmetric space of the non-compact type. This class consists of the groups with finite centre associated to the Lie algebras $\mathfrak{su}(p,q)$, $\mathfrak{sp}(2n,\mathbb{R})$, $\mathfrak{so}^*(2n)$, $\mathfrak{so}(2,n)$, $\mathfrak{e}_6^{-14}$ and $\mathfrak{e}_7^{-25}$. In the exceptional Lie algebras, the superindex refers to the signature of the Killing form (see, e.g., [Hel01]). When G is Hermitian, the induced almost complex structure splits $\mathfrak{m}^{\mathbb{C}}$ into $(\pm i)$-eigenspaces $\mathfrak{m}^+$ and $\mathfrak{m}^-$. A G-Higgs field then has two components $\beta \in H^0(E(\mathfrak{m}^+)\otimes K)$ and $\gamma \in H^0(E(\mathfrak{m}^-)\otimes K)$. Since the centre of H is a circle, then $\pi_1 H$ is isomorphic to $\mathbb{Z}$ plus possibly a torsion part (which only appears in $\mathfrak{so}(2,n)$). The projection to $\mathbb{Z}$ defines an invariant which we prove to be bounded. The bundles whose invariant attains the maximal (positive or negative) value are called maximal G-Higgs bundles. As in the case of representations, this invariant separates $\mathcal{M}(G)$ into subvarieties. The main goal of this thesis is to study the moduli space $\mathcal{M}_{max}(G)$ of polystable G-Higgs bundles with maximal value of this invariant, when G is a non-compact semisimple Lie group of Hermitian type with finite centre.

Although not in the context of moduli spaces, this invariant was studied in [Mil58], where Milnor related the Euler class of a $\mathrm{GL}^+(2,\mathbb{R})$-bundle over X with the existence of a flat connection over X. He proved that such a connection exists if and only if the Euler class e satisfies $|e| \leq g-1$. In fact, this Euler class coincides with the invariant above by the projection $\pi_1 G$ to $\mathbb{Z}$. This result was generalized by Wood in [Woo71] to the groups of homeomorphisms and diffeomorphisms of the circle, which justifies the name given to this inequality. In [Gol80], [Gol82] and [Gol88], Goldman studied the space of representations $\mathrm{Hom}(\pi_1 X, \mathrm{PSL}(2,\mathbb{R}))$ and counted the number of connected components. Goldman used the Euler number as invariant and proved that representations with maximal invariant are faithful and discrete. He identified the subvariety of maximal representations with the Teichmüller space. These components had previously drawn the attention of Weil when he studied the space of representations of an abstract group in a topological group ([Wei60], [Wei62]).

Our Higgs bundle approach follows the work of Hitchin ([Hit87]) for $\mathrm{SL}(2,\mathbb{R}) \cong \mathrm{SU}(1,1)$. An $\mathrm{SL}(2,\mathbb{R})$-Higgs bundle is equivalent to a holomorphic vector bundle $V = L \oplus L^{-1}$, where L is a line bundle, together with a Higgs field $\varphi \in H^0(X, \mathrm{End}_0 V \otimes K)$ consisting of two components, $\beta : L \to L^{-1}\otimes K$ and $\gamma : L^{-1} \to L \otimes K$, where K denotes the canonical line bundle over X.

$$\varphi = \begin{pmatrix} 0 & \gamma \\ \beta & 0 \end{pmatrix} \in H^0(X, \mathrm{End}_0 V \otimes K).$$

In this case, the bound of the invariant is a consequence of the semistability condition of the Higgs bundle. Hitchin also paid special attention to the objects with maximal invariant. He proved that for a (positive) maximal stable Higgs bundle, L must be a square root of K, the field β must be the identity and the field γ is a quadratic differential, i.e., $\gamma \in H^0(X, K^2)$. Furthermore, he identified the maximal component with the Teichmüller space. He later defined Hitchin-Teichmüller components for split real forms of simple complex Lie groups in [Hit92], which include the group $\mathrm{Sp}(2n, \mathbb{R})$, both split and Hermitian, and for which Hitchin-Teichmüller components are indeed maximal.

The fact that for maximal Higgs bundles we have $L = K^{1/2}$, $\beta = \mathrm{id}$ and $\gamma \in H^0(X, K^2)$, and the conditions on maximal representations (faithfulness and discreteness) revealed a phenomenon of rigidity for the maximal objects which has been widely studied. Instead of the integer invariant defined above as the projection of the topological class to $\mathbb{Z}$, we will use the Toledo invariant. As shown in Section 3.5, the former is simply an integer multiple of the Toledo invariant, which may be rational. Toledo defined in [Tol89] an invariant for a representation $\rho : \pi_1 X \to \mathrm{PU}(1, n)$ as follows. The group $\mathrm{PU}(1, n)$ is the isometry group of the hyperbolic n-space $\mathbf{H}^n_{\mathbb{C}}$, which is identified with the symmetric space $\mathrm{PU}(1, n)/\mathrm{P}(U(1) \times U(n))$. Consider the Kähler form ω corresponding to a Hermitian (normalized) metric of minimal holomorphic sectional curvature -1, the Bergmann metric. The representation $\rho : \pi_1 X \to \mathrm{PU}(1, n)$ determines a ρ-equivariant developing map $f : \widetilde{X} \to \mathbf{H}^n_{\mathbb{C}}$. The pullback $f^*\omega$ defines a $\pi_1 X$-invariant form on $\widetilde{X}$ and hence descends to a form on X which we also denote by $f^*\omega$. The Toledo invariant is then defined as

$$T(\rho) = \int_X f^*\omega.$$

This process can be generalized replacing the hyperbolic n-space by any non-compact Hermitian symmetric space G/H with its Bergmann metric. Domic and Toledo ([DT87]) had already proved that the Toledo invariant satisfied a Milnor-Wood type inequality by finding a bound to the norm of the Kähler class of the symmetric space. When this invariant attains its maximal values, Toledo detected a certain rigidity consisting in the stabilization of a complex geodesic, and hence the image of maximal representations lies in the subgroup $\mathrm{P}(\mathrm{U}(1, 1) \times \mathrm{U}(n - 1))$. The complex geodesic corresponds to a maximal tube-type subspace of $\mathbf{H}^n_{\mathbb{C}}$. This work was generalized by Hernández in [Her91] for the group $\mathrm{PSU}(2, q)$. The generalization for $\mathrm{PU}(p, q)$ was obtained using Higgs bundles techniques by Bradlow, García-Prada and Gothen in [BGG01]. This work has been followed by the study of $\mathrm{U}(p, q)$, which includes

$\mathrm{SU}(p,q)$, in [BGG03] and $\mathrm{SO}^*(4n+2)$ ([BGG06], [BGG12]). A different and general approach using bounded cohomology has been used by Burger, Iozzi, Wienhard, as it is explained in [BIW10b]. This approach also allows one to define the Toledo invariant also for surfaces with boundary.

As we have already mentioned above, in this thesis, we study the moduli space of polystable G-Higgs bundles over a Riemann surface X, when G is a simple non-compact real Lie group of Hermitian type with finite centre. Our main results are concerned with finding a bound of the Toledo invariant and describing rigidity phenomena inherent to maximal G-Higgs bundles, those whose Toledo invariant attains one of the bounds. We generalize the work described above to arbitrary groups of Hermitian type and provide a classification-independent proof based on how the geometry of the Hermitian symmetric space associated with G is reflected in the moduli space. In order to do this, we take a new approach to the Toledo invariant, by defining a character that we call the Toledo character.

In Chapter 2, we present some basic facts about Hermitian symmetric spaces, following [Hel01], [FKK+00] and [Ji05]. Since the aim is the study of G-Higgs bundles, we focus also on the group G apart from the standard focus on the corresponding symmetric space G/H. The finite coverings or quotients of a given Hermitian group all give the same Hermitian symmetric space, but the G-Higgs bundles associated to this group are different objects. The Harish-Chandra embedding realizes G/H as a bounded symmetric domain in $\mathfrak{m}^+$ and the generalized Cayley transform gives an alternative realization as a generalized half-plane ([KW65]). This half-plane may be biholomorphic to a tube domain over a cone Ω, in which case we say that G is of tube-type. Otherwise, we say that G is of non-tube type. Following the classification of irreducible symmetric spaces of Cartan ([Car26], [Car27]), one has that the groups corresponding to $\mathfrak{sp}(2n,\mathbb{R})$, $\mathfrak{so}^*(4m)$, $\mathfrak{so}(2,n)$, $\mathfrak{su}(p,p)$, and $\mathfrak{e}_7^{-25}$ are of tube type, whilst $\mathfrak{so}^*(4m+2)$, $\mathfrak{su}(p,q)$ for $p\neq q$ and $\mathfrak{e}_6^{-14}$ are of non-tube type. This realization gives the ingredients with which the main results are stated. On the one hand, for the groups of tube-type, $\mathfrak{m}^+$ has the structure of unital Jordan algebra (with unit labelled as e_Γ, see Section 2.4.2) and determinant det, and the cone Ω is the symmetric space H^*/H_0', where H^* is the non-compact dual of H and H_0' is the identity component of the stabilizer of e_Γ in H^*. On the other hand, every group of non-tube type has a maximal subgroup of tube type. In particular, this allows one to define a notion of rank on $\mathfrak{m}^+$, even though the determinant does not exist. This can be done similarly for $\mathfrak{m}^-$.

The tube-type condition is also reflected in the root system of $\mathfrak{g}$. Since G is Hermitian, a Cartan subalgebra $\mathfrak{t}^{\mathbb{C}}$ can be chosen in $\mathfrak{h}^{\mathbb{C}}$, so that every root $\alpha \in \Delta(\mathfrak{g}^{\mathbb{C}}, \mathfrak{t}^{\mathbb{C}})$ satisfies that its root space $\mathfrak{g}_{\alpha}^{\mathbb{C}}$ lies either in $\mathfrak{h}^{\mathbb{C}}$, and we call the root compact (and write $\alpha \in \Delta_C$), or in $\mathfrak{m}^{\mathbb{C}}$, and we call the root non-compact (and write $\alpha \in \Delta_Q$). The roots in $\Delta(\mathfrak{g}^{\mathbb{C}}, \mathfrak{t}^{\mathbb{C}})$ are restricted to a certain subalgebra $i\mathfrak{t}^{-} \subset \mathfrak{t}^{\mathbb{C}}$ given by a choice of non-compact roots. The resulting elements form a, not-necessarily reduced, root system, called the restricted root system, which is studied in Section 2.3. This system can only be of two types: reduced which corresponds to tube-type groups, and non-reduced which corresponds to non-tube-type groups. Some roots may project to the same restricted root, and the possible multiplicities of a restricted root are invariants of the system. In terms of these multiplicities, the dual Coxeter number N, an invariant associated to the Lie algebra $\mathfrak{g}$, is defined in Section 2.3.1.

A root system coming from a non-tube group G always contains a maximal tube-type root system, which gives the subalgebra $\mathfrak{g}_T^{\mathbb{C}} \subset \mathfrak{g}^{\mathbb{C}}$, the subalgebra $\mathfrak{h}_T^{\mathbb{C}} \subset \mathfrak{h}^{\mathbb{C}}$ with its corresponding subgroup $H_T^{\mathbb{C}} \subset H^{\mathbb{C}}$, and the vector space $\mathfrak{m}_T$.

The Toledo character χ_T is defined in Section 2.4. It is defined as a character of the Lie algebra $\mathfrak{h}^{\mathbb{C}}$ by using the root system of $\mathfrak{g}^{\mathbb{C}}$. The sum of the positive non-compact roots is an element of the dual of $\mathfrak{t}^{\mathbb{C}}$, of which we consider the rational multiple

$$\chi_T = \frac{2}{N} \sum_{\alpha \in \Delta_Q^+} \alpha.$$

Equivalently, the dual s_{χ_T} of χ_T with respect to the Killing form is given by an element of $\mathfrak{t}^{\mathbb{C}}$. Lemma 2.37 shows that this element is indeed in the centre, $s_{\chi_T} \in i\mathfrak{z} \subset \mathfrak{z}^{\mathbb{C}} \cong \mathfrak{h}^{\mathbb{C}}/[\mathfrak{h}^{\mathbb{C}}, \mathfrak{h}^{\mathbb{C}}]$, and hence defines a character of $\mathfrak{h}^{\mathbb{C}}$. The exponentiation of this character to the group $H^{\mathbb{C}}$ depends on four invariants. Two coming from the Lie algebra $\mathfrak{g}$: the dimension of $\mathfrak{m}$ and the dual Coxeter number. And two coming from the group G: the finite number $l = |Z_0^{\mathbb{C}} \cap [H^{\mathbb{C}}, H^{\mathbb{C}}]|$, where Z_0 is the identitity component of the centre of $H^{\mathbb{C}}$, and the order of the exponentiation of the element $2\pi J$ as element of G, where $J \in \mathfrak{z}(\mathfrak{h})$ is the element giving the almost complex structure on $\mathfrak{m}$ by its adjoint action. Proposition 2.39 gives the numerical condition for the character $q \cdot \chi_T$ to exponentiate to $H^{\mathbb{C}}$:

$$\frac{q \cdot \dim \mathfrak{m} \cdot o(e^{2\pi J})}{l \cdot N} \in \mathbb{Z}.$$

The unit element of the Jordan algebra, $e_\Gamma \in \mathfrak{m}^{+}$, defines a parabolic subalgebra of $\mathfrak{h}^{\mathbb{C}}$ given by

$$\mathfrak{p}_{e_\Gamma} = \operatorname{Ker}(\operatorname{ad}(e_\Gamma)_{|\mathfrak{h}^{\mathbb{C}}}) \oplus \operatorname{Im}(\operatorname{ad}(e_\Gamma)_{|\mathfrak{m}^{-}}),$$

for which the Toledo character is an antidominant character, as Proposition 2.60 shows. This statement is extended by conjugation to any element of non-zero determinant in $\mathfrak{m}^+$. The proof of these statements requires some technical results involving restricted root systems which are proved in Section 2.3.

For groups of tube type, one has that a rational multiple of the Toledo character $q_T \cdot \chi_T$ lifts to a character of the group, $\widetilde{\chi}_T$, and this new character describes the semi-invariance of the determinant on $\mathfrak{m}^+$ with respect to the action of $H^{\mathbb{C}}$, namely

$$\det(h \cdot x)^{q_T} = \widetilde{\chi}_T(h) \det(x)^{q_T},$$

for $h \in H^{\mathbb{C}}$ and $x \in \mathfrak{m}^+$. Thus, the action of $H^{\mathbb{C}}$ preserves the elements of non-zero determinant, $\mathfrak{m}^+_{D\neq 0}$. Moreover, $H^{\mathbb{C}}$ acts transitively on these elements and $\mathfrak{m}^+_{D\neq 0}$ is the homogeneous space $H^{\mathbb{C}}/H'^{\mathbb{C}}$, where H' is the stabilizer in H of the element e_Γ.

The study of α-polystable G-Higgs bundles (E, β, γ) for G of Hermitian type and α a parameter in the centre of $\mathfrak{h}^{\mathbb{C}}$ begins in Chapter 3. The notion of α-polystability is given in Definition 3.5 in terms of reductions of the structure group of E from $H^{\mathbb{C}}$ to Richardson parabolic subgroups (studied in Section 2.5) and their antidominant characters. The Toledo invariant is defined from the Toledo character as

$$d = \frac{1}{q_T} \deg(E(\widetilde{\chi}_T)).$$

This definition agrees with the previous definition of the Toledo invariant referred above, as shown in Section 5.2. Moreover, from the notion of rank of the Jordan algebra $\mathfrak{m}^+$, and also $\mathfrak{m}^-$, we define a rank for the components of the Higgs field $\beta \in H^0(E(\mathfrak{m}^+) \otimes K)$ and $\gamma \in H^0(E(\mathfrak{m}^-) \otimes K)$.

The Milnor-Wood type inequality proved in Section 3.4 gives a bound for the Toledo invariant d of a semistable G-Higgs bundle (E, β, γ) based on the parameter and the ranks of the two components of the field. Namely, one has the following.

Theorem (3.18). *Let $\alpha \in i\mathfrak{z}$ such that $\alpha = i\lambda J$ for $\lambda \in \mathbb{R}$. Let (E, β, γ) be an α-semistable G-Higgs bundle. Then, the Toledo invariant $d = \frac{1}{q_T} \deg(E(\widetilde{\chi}_T))$ satisfies:*

$$-\operatorname{rk}(\beta)(2g-2) - \left(\frac{2\dim\mathfrak{m}}{N} - \operatorname{rk}(\beta)\right)\lambda \leq d \leq \operatorname{rk}(\gamma)(2g-2) + \left(\frac{2\dim\mathfrak{m}}{N} - \operatorname{rk}(\gamma)\right)\lambda,$$

where N is the dual Coxeter number and $\dim\mathfrak{m}$ is the dimension of the isotropy representation of G. In the tube-type case, this simplifies to:

$$-\operatorname{rk}(\beta)(2g-2) - (r - \operatorname{rk}(\beta))\lambda \leq d \leq \operatorname{rk}(\gamma)(2g-2) + (r - \operatorname{rk}(\gamma))\lambda.$$

The proof provided here is independent of the classification theorem and is based on the semistability condition and the structure of Jordan algebra of a subtube.

As a consequence, we obtain the Milnor-Wood inequality.

Theorem (4.1)**.** *Let G be a simple group of Hermitian type. Let d be the Toledo invariant of a semistable G-Higgs bundle. Then,*

$$|d| \leq \operatorname{rk}(G/H)(2g-2).$$

Let the parameter α be zero from now on. This case is of special interest by the connection with surface group representations described above. Nonetheless, the study of parameter-depending stability may be of help when using Morse theory to count the number of connected components of the moduli space, as shown in [BGG03].

Chapter 4 is devoted to the study of the moduli space of maximal, i.e., $d = \pm \operatorname{rk}(G/H)(2g-2)$, polystable G-Higgs bundles $\mathcal{M}_{max}(G)$ and contains the main results of the thesis, which describe the rigidity of maximal G-Higgs bundles.

When G is of tube type, $\mathcal{M}_{max}(G)$ imbeds into the moduli space of polystable K^2-twisted H^*-Higgs pairs, where H^* is the non-compact dual of H which gives the cone $\Omega \cong H^*/H_0'$ related to G/H.

Theorem (4.7). *Let G be a Hermitian group of tube type and H be a maximal compact subgroup. Let H^* be the non-compact dual of H in $H^{\mathbb{C}}$. Let J be the element in the centre of the Lie algebra $\mathfrak{g}$ giving the almost complex structure on $\mathfrak{m}$ (see Proposition 2.2). If the order of $e^{2\pi J} \in H^{\mathbb{C}}$ divides $(2g-2)$, then there is an injection of complex algebraic varieties*

$$\mathcal{M}_{\max}(G) \to \mathcal{M}_{K^2}(H^*).$$

Moreover, stable G-Higgs bundles correspond to stable K^2-twisted H^*-Higgs pairs.

We believe that this is actually an isomorphism as shown for the $\mathrm{SU}(p,p)$, $\mathrm{Sp}(2n,\mathbb{R})$ and $\mathrm{SO}^*(4m)$ in [BGG03], [GGM08] and [BGG12], respectively.

When G is of non-tube type, we obtain the following rigidity theorem.

Theorem (4.15). *Let G be a simple Hermitian group of non-tube type and let H be its maximal compact subgroup. Then, there are no stable G-Higgs bundles with maximal Toledo invariant. In fact, every polystable maximal G-Higgs bundle reduces to a stable $N_G(\mathfrak{g}_T)_0$-Higgs bundle, where $N_G(\mathfrak{g}_T)_0$ is the identity component of the normalizer of $\mathfrak{g}_T$ in G.*

Thus, the structure group of maximal G-Higgs bundles is reduced to a strictly smaller group than G. Therefore, the dimension of the moduli space is smaller than the expected dimension.

Moreover, the second part of Section 4.3 is devoted to the moduli space of maximal polystable G-Higgs bundles and its realization as a fibration of other moduli spaces. The technical results needed to do this is developed in Section 2.6.

These rigidity phenomena have been widely studied, also from the point of view of representations. The study of maximal representations has attracted much interest because of its geometric significance. In the case of $\mathrm{SL}(2,\mathbb{R})$, Goldman ([Gol80]) proved that there are 2^{2g} maximal components in the moduli space of representations, which can be identified with the Teichmüller space, and they consist entirely of discrete and faithful representations. Using methods of bounded cohomology, Burger, Labourie, Iozzi and Wienhard have proved in general that the maximal components for Hermitian groups consist entirely of discrete and faithful representations. An interesting result coming from this approach is that maximal representations are necessarily reductive, so the hypothesis of reductivity for Higgs bundles is thus satisfied in the maximal case.

For tube-type groups, the Cayley correspondence $\mathcal{M}_{max}(G) \to \mathcal{M}_{K^2}(H^*)$ proved in Theorem 4.7, shows the rigidity of maximal objects. The corresponding K^2-twisted H^*-pair has a structure group smaller than the initial G-Higgs bundle. Although it is not a reduction of the structure group, the dimension of the new group H^* equals the dimension of a maximal compact subgroup H of G. Furthermore, this correspondence reveals new invariants for maximal G-Higgs bundles and surface group representation into G. These are the invariants coming from the group H^*. For example, when $G = \mathrm{Sp}(2n,\mathbb{R})$, we have that $H^* = \mathrm{GL}(n,\mathbb{R})$ with $H' = \mathrm{O}(n)$ as a maximal compact subgroup. To a $\mathrm{Sp}(2n,\mathbb{R})$-Higgs bundle we can thus attach the first and second Stiefel-Whitney classes of the corresponding $\mathrm{GL}(n,\mathbb{R})$-Higgs pair via the Cayley correspondence, $w_1 \in H^1(X,\mathbb{Z}/2)$, $w_2 \in H^2(X,\mathbb{Z}/2)$. These invariants provide a way to count the connected components of the moduli space $\mathcal{M}_{max}(\mathrm{Sp}(2n,\mathbb{R}))$, as surveyed in [Got11]. These invariants have appeared from the point of view of representations in the work of Guichard and Wienhard, [GW09], where they define analogues of w_1, w_2 for $(\mathrm{Sp}(2n,\mathbb{R}), \mathrm{GL}(n,\mathbb{R}))$-Anosov representations, of which maximal representations are particular cases.

For the exceptional cases we have the following theorems.

Theorem (4.13)**.** *There exists an imbedding of complex algebraic varieties*

$$\mathcal{M}_{max}(\mathrm{E}_7^{-25}) \to \mathcal{M}_{K^2}(\mathrm{E}_6^{-26} \ltimes \mathbb{R}^*)$$

Theorem (4.17)**.** *Every maximal E_6^{-14}-Higgs bundle is strictly polystable and reduces to a stable* $\mathrm{Spin}_0(2,8) \times \mathrm{U}(1)$*-Higgs bundle and hence it is a product of a* $\mathrm{Spin}_0(2,8)$*-Higgs bundle and a line bundle. Moreover, the principal* $\mathrm{Spin}_0(2,8)$*-Higgs bundle is maximal.*

Chapter 5 deals with the relation of Higgs bundles with representations. We start by sketching the proof of the equivalence of the moduli space of polystable G-Higgs bundles over X with the moduli space of representations of $\pi_1 X$ into G. We then show the equivalence of several notions of Toledo invariant. In Section 5.3 we take advantage of the correspondence $\mathcal{R}(\pi_1 X, G) \cong \mathcal{M}(G)$ to state the main results in terms of representations, and review previous and different approaches to the rigidity of maximal representations. For non-tube-type groups we have the following theorem.

Theorem (5.4)**.** *Let $\rho : \pi_1 X \to G$ be a maximal representation of the fundamental group of a Riemann surface X into a semisimple Hermitian Lie group of non-tube type G. Then, the image of ρ is contained in $N_G(\mathfrak{g}_T)_0$, where $\mathfrak{g}_T$ is the subalgebra of $\mathfrak{g}$ corresponding to a maximal tube type subdomain G_T/H_T of G/H.*

As a consequece of this theorem we have the following.

Theorem (5.5)**.** *The image of any maximal representation $\rho : \pi_1 X \to E_6^{-14}$ is contained in* $\mathrm{Spin}_0(2,8) \times \mathrm{U}(1)$*.*

The thesis ends with three appendices. In the first appendix, the Cartan data and the restricted root system is described for the classical and exceptional Hermitian Lie algebras. The second appendix contains some additional remarks, while the third one shows several tables about the Cayley ingredients and Higgs bundles.

Chapter 2

Hermitian symmetric spaces

2.1 Basics on symmetric spaces and root theory

A Riemannian manifold (M, g) is said to be **symmetric** if for every point $p \in M$ there exists a global isometry s_p which is an involution and only fixes p. The identity component of the group of isometries, $G := I(M)_0$, endowed with the compact-open topology becomes a Lie group by the Myers-Steenrod theorem ([MS39]). As a consequence of being symmetric, it acts transitively on M. Choosing a point $o \in X$, the isotropy group $H = Stab_G(o)$ is a compact subgroup and there is an analytic isomorphism $M \cong G/H$ defined by $g \cdot o \mapsto gH$.

We define an involution on G by $g \mapsto s_o g s_o$ and consider its differential $\theta : \mathfrak{g} \to \mathfrak{g}$, which is also an involution, $\theta^2 = Id$. The $(+1)$-eigenspace $\mathfrak{h}$ is the Lie algebra of H and the (-1)-eigenspace $\mathfrak{m}$ is a vector space isomorphic to the tangent space at o, T_oM, via the map $Y \mapsto Y \cdot o = \frac{d}{dt}|_{t=0} \exp(tY) \cdot o$. By defining $Q(Y, Y') = g(Y \cdot o, Y' \cdot o)$ we get an $Ad(H)$-invariant positive definite quadratic form on $\mathfrak{m}$. The eigenspaces satisfy $[\mathfrak{h}, \mathfrak{h}] \subset \mathfrak{m}$, $[\mathfrak{h}, \mathfrak{m}] \subset \mathfrak{m}$ and $[\mathfrak{m}, \mathfrak{m}] \subset \mathfrak{h}$. For $\mathfrak{g}$ semisimple, $\mathfrak{g} = \mathfrak{h} + \mathfrak{m}$ is a Cartan decomposition and θ is a Cartan involution.

The pair $(\mathfrak{g}, \theta)$ is endowed with a θ and $\mathrm{ad}(\mathfrak{h})$-invariant metric. Pairs consisting of an algebra and an involution with such a metric are called orthogonal involutive Lie algebras and they all come from some symmetric space. However, the pair does not determine uniquely a symmetric space, but a family of covers and quotients. Let $\widetilde{G}$ be the simply connected group with Lie algebra $\mathfrak{g}$ and $\widetilde{\theta}$ the lifting of the involution to $\widetilde{G}$. Denote by $\widetilde{H}$ the fixed point set for $\widetilde{\theta}$. Any subgroup H such that $\widetilde{H}_0 \subset H \subset \widetilde{H}$ satisfies the property that $\widetilde{G}/H$ is a symmetric space, which is covered by the simply connected one $\widetilde{G}/\widetilde{H}_0$. Note that $\tilde{G}$ may not be the isometry group.

Example 2.1. *The group of isometries of the Poincaré Disk* $\{z \in \mathbb{C} \mid |z| \leq 1\}$ *is*

$$G = \mathrm{SU}(1,1) = \left\{ \begin{pmatrix} \alpha & \beta \\ \bar{\beta} & \bar{\alpha} \end{pmatrix} \mid \alpha, \beta \in \mathbb{C},\ \alpha\bar{\alpha} - \beta\bar{\beta} = 1 \right\},$$

acting by linear transformations $z \mapsto \frac{\alpha z + \beta}{\bar{\beta} z + \bar{\alpha}}$. *The isotropy group at* 0 *is*

$$H = \left\{ \begin{pmatrix} \alpha & 0 \\ 0 & \bar{\alpha} \end{pmatrix} \mid \alpha \in \mathbb{C},\ \alpha\bar{\alpha} = 1 \right\} \cong U(1).$$

The Cartan decomposition is $\mathfrak{su}(1,1) = \mathfrak{u}(1) + \mathfrak{m}$, *i.e.,*

$$\begin{pmatrix} ia & z \\ -\bar{z} & -ia \end{pmatrix} = \begin{pmatrix} ia & 0 \\ 0 & -ia \end{pmatrix} + \begin{pmatrix} 0 & z \\ -\bar{z} & 0 \end{pmatrix}.$$

A pair $(\mathfrak{g}, \theta)$ is called Euclidean if $[\mathfrak{m}, \mathfrak{m}] = 0$; reduced if the isotropy representation $\mathrm{ad} : \mathfrak{h} \to \mathrm{End}(\mathfrak{m})$ is faithful, or equivalently, if $\mathfrak{h}$ does not contain any non-zero ideal of $\mathfrak{g}$; and irreducible if $\mathrm{ad} : \mathfrak{h} \to \mathrm{End}(\mathfrak{m})$ is an irreducible representation. Any reduced pair can be decomposed as a sum $\mathfrak{g} = \mathfrak{g}_1 \oplus \ldots \oplus \mathfrak{g}_n$ of θ-stable subalgebras. One of them may be Euclidean, while the others are called non-Euclidean and are irreducible and reduced. These non-Euclidean factors are either of the so-called compact type, corresponding to $\mathfrak{g}_j$ compact, or non-compact type, corresponding to $\mathfrak{g}_j$ non-compact.

The pair $(\mathfrak{g}, \theta)$ coming from the isotropy group of a symmetric space M is reduced. The space is called non-Euclidean (resp. compact, non-compact) if all the factors in the decomposition of $\mathfrak{g}$ are non-Euclidean (resp. compact, non-compact). From now on, we consider all our pairs and spaces to be non-Euclidean.

We define the dual of an irreducible pair $(\mathfrak{g}, \theta)$ by $\mathfrak{g}^* = \mathfrak{h} + i\mathfrak{m} \subset \mathfrak{g}^{\mathbb{C}}$ and the involution given by extending θ to $\mathfrak{g}^{\mathbb{C}}$ and restricting then to $\mathfrak{g}^*$. One of the pairs $(\mathfrak{g}, \theta)$, $(\mathfrak{g}^*, \theta)$ is compact and the other is non-compact.

A pair of groups (G, H) is said to be associated to the orthogonal Lie algebra $(\mathfrak{g}, \theta)$ if G is a connected Lie group with Lie algebra $\mathfrak{g}$ and H is a connected Lie group with Lie algebra $\mathfrak{h}$. If $(\mathfrak{g}, \theta)$ is non-compact, we have that H is connected, closed and contains the centre of G. Moreover, H is compact if and only if the centre of G is finite, and in this case, H is a maximal compact subgroup of G ([Hel01], Ch. VI, Th. 1.1). By the connectedness of H we have that there is only one non-compact symmetric space associated to $(\mathfrak{g}, \theta)$, which is indeed simply connected. This may not be the case for pairs of compact type. The Killing form of $\mathfrak{g}$, denoted by B as usual, is negative definite in $\mathfrak{h}$ and positive definite in $\mathfrak{m}$. So we define a positive definite quadratic form by $B_\theta = -B(X, \theta Y)$.

A symmetric space is said to be irreducible if it can not be expressed as a product of non-trivial symmetric spaces. In this case, $\mathfrak{g}$ is in general semisimple, and simple in the non-compact case, since every simply connected symmetric space can be written as a product of irreducible symmetric spaces.

A symmetric space is said to be **Hermitian** if it has a complex structure compatible with the metric, and each involution s_p is a holomorphic isometry. Due to its invariance under isometries, this complex structure is always Kähler.

From now on, we consider non-compact irreducible Hermitian symmetric spaces. These are given by quotients G/H of a connected non-compact simple Lie group G with trivial centre by a maximal compact subgroup H with non-discrete centre ([Hel01], Ch. VIII, Th. 6.1). This result is obtained by passing from the symmetric space as a quotient of simply connected groups $\widetilde{G}/\tilde{H}$ to the quotient $\mathrm{Ad}_{\widetilde{G}}(\widetilde{G})/\mathrm{Ad}_{\widetilde{G}}(\tilde{H})$.

Our motivation is the study of G-Higgs bundles. Since the definition of G-Higgs bundle involves a maximal compact subgroup of G (as we will see in Section 3.1), we will only consider groups G such that H is a maximal compact subgroup, i.e., groups G with finite centre. These groups are finite covers of the adjoint group $\mathrm{Ad}_{\widetilde{G}}(\widetilde{G}) = \mathrm{Ad}_G(G)$, which we denote by $\mathrm{Ad}\, G$. The following diagram is satisfied for $\Gamma = \mathrm{Ker}\, \pi$.

$$\begin{tikzcd}
1 \arrow[r] & \Gamma \arrow[r] \arrow[d, equal] & H \arrow[r, "\pi"] \arrow[d, hook] & \mathrm{Ad}_G H \arrow[r] \arrow[d, hook] & 1 \\
1 \arrow[r] & \Gamma \arrow[r] & G \arrow[r, "\pi"] & \mathrm{Ad}\, G \arrow[r] & 1
\end{tikzcd}$$

From the theory of covering spaces, we know that for $\widetilde{G} \to G \to \mathrm{Ad}\, G$ we have that $Z(\widetilde{G}) \cong \pi_1(\mathrm{Ad}\, G)$, $G \cong \frac{\widetilde{G}}{\Gamma}$ for $\Gamma \subset Z(\widetilde{G})$, $\pi_1(G) \cong \Gamma$ and $Z(G) \cong \frac{Z(\widetilde{G})}{\Gamma}$.

Proposition 2.2. *The pair $(\mathfrak{g}, \mathfrak{h})$ associated to a Hermitian symmetric space G/H satisfies the condition that $\mathfrak{h}$ is reductive ($\mathfrak{h} = \mathfrak{z} + [\mathfrak{h}, \mathfrak{h}]$, with centre $\mathfrak{z}$). In the irreducible case, $\mathfrak{z}$ is one-dimensional and one of its generators $J \in \mathfrak{z}$ gives the almost complex structure on $\mathfrak{m}$ by $J_0 = \mathrm{ad}(J)_{|\mathfrak{m}}$.*

Since $J_0^2 = -\mathrm{Id}$, we decompose $\mathfrak{m}^{\mathbb{C}}$ into $\pm i$-eigenspaces for J_0:

$$\mathfrak{m}^{\mathbb{C}} = \mathfrak{m}^+ + \mathfrak{m}^-.$$

These eigenspaces are Abelian, $[\mathfrak{m}^+, \mathfrak{m}^+] = [\mathfrak{m}^-, \mathfrak{m}^-] = 0$, and $\mathfrak{h}^{\mathbb{C}}$ acts on them, $[\mathfrak{h}^{\mathbb{C}}, \mathfrak{m}^{\pm}] \subset \mathfrak{m}^{\pm}$. The lifting of this action is called the isotropy representation

$$\mathrm{Ad} : H^{\mathbb{C}} \to \mathrm{Aut}(\mathfrak{m}^{\mathbb{C}}).$$

Remark 2.3. If we consider the group $\mathrm{Ad}_G(G)$ with maximal compact subgroup $\mathrm{Ad}_G(H)$, the isotropy representation is faithful, but this is not always the case (see [Bes08], p.179).

We prove a simple lemma about the centre of G.

Lemma 2.4. *The centre $Z(G)$ is contained in $Z(H)$.*

Proof. Let $he^X \in G = H \exp \mathfrak{m}$, an element of the centre. From the commutativity with H, one gets that both h and e^X must lie in $Z(H)$. But $e^X \in Z(H)$ is only possible if $X = 0$, since $[\mathfrak{h}, \mathfrak{m}] = \mathfrak{m}$. □

The order of the element $e^{2\pi J} \in Z(G) \subset Z(H)$ plays an important role in the following lemma.

Lemma 2.5. *In the irreducible case, there exists an isomorphism $\mu : \mathbb{C}^* \cong Z(H^{\mathbb{C}})$ such that the centre of $H^{\mathbb{C}}$ acts on $\mathfrak{m}^+$ by*

$$\begin{aligned} \mathrm{Ad} \circ \mu : & \mathbb{C}^* \xrightarrow{\mu} Z(H^{\mathbb{C}}) \to \mathrm{Aut}(\mathfrak{m}^+) \\ & \lambda \mapsto \mu(\lambda) \mapsto \lambda^{o(e^{2\pi J})} \cdot \mathrm{Id}\,. \end{aligned}$$

Proof. Let $s = o(e^{2\pi J})$. The centre $Z(H^{\mathbb{C}})$ equals $\{e^{aJ}\}_{a \in \mathbb{C}}$, with $J \in \mathfrak{z}(\mathfrak{h}^{\mathbb{C}})$ giving the almost complex structure. We consider the infinite covering $\exp : \mathbb{C} \to \mathbb{C}^*$ and define $d\mu : \mathbb{C} \to \mathfrak{z}(\mathfrak{h}^{\mathbb{C}})$ by $ai \mapsto asJ$. This map lifts to $\mu : \mathbb{C}^* \to Z(H^{\mathbb{C}})$, since for a such that $e^{ai} = 1$, i.e. $a \in 2\pi\mathbb{Z}$, we have that e^{asJ} is the identity in $H^{\mathbb{C}}$. The action of $\mathrm{Ad} \circ \mu$ then follows from the diagram

$$\begin{tikzcd} ai \arrow[r, "d\mu"] \arrow[d, "\exp"] & asJ \arrow[r, "\mathrm{ad}"] \arrow[d, "\exp"] & asi\,\mathrm{Id} \arrow[d, "\exp"] \\ \lambda = e^{ai} \arrow[r, "\mu"] & e^{asJ} \arrow[r, "\mathrm{Ad}"] & \lambda^s\,\mathrm{Id} \end{tikzcd} .$$

□

Lemma 2.6. *There are $\mathrm{Ad}(H)$-equivariant isomorphisms between $\mathfrak{m}^+$, $\mathfrak{m}$ and $\mathfrak{m}^-$, regarded as vector spaces, given by*

$$\begin{tikzcd} \mathfrak{m}^+ \arrow[r, "\varphi_+"] & \mathfrak{m} & \mathfrak{m}^- \arrow[l, "\varphi_-"'] \end{tikzcd} .$$

$$\tfrac{1}{2}(X - iJ_0X) \longleftarrow\!\!\dashv X \longmapsto \tfrac{1}{2}(X + iJ_0X)$$

We denote by φ_+^- the map $\varphi_+ \circ \varphi_-^{-1} : \mathfrak{m}^+ \cong \mathfrak{m}^-$ and similarly $\varphi_-^+ = \varphi_- \circ \varphi_+^{-1}$. These two isomorphisms are in fact $\mathrm{Ad}(H^{\mathbb{C}})$-equivariant. The non-discrete centre of $H^{\mathbb{C}}$ regarded as $\mathbb{C}^*$ acts in $\mathfrak{m}^+$ as follows.

We review the embedding theorems for a non-compact Hermitian symmetric space G/H. Let $\mathfrak{g} = \mathfrak{h} + \mathfrak{m}$ be the Cartan decomposition and $\mathfrak{u} = \mathfrak{h} + i\mathfrak{m}$ be its compact dual. We denote by τ the involution of $\mathfrak{g}^{\mathbb{C}}$ which fixes $\mathfrak{u}$. Let G^C be the simply connected Lie group with Lie algebra $\mathfrak{g}^{\mathbb{C}}$. For the subalgebras $\mathfrak{g}$, $\mathfrak{u}$, $\mathfrak{h}$, $\mathfrak{h}^{\mathbb{C}}$, $\mathfrak{m}^+$ and $\mathfrak{m}^-$ there exist corresponding subgroups of G^C: G^0, U, H^0, $H^{0,\mathbb{C}}$, M^+ and M^-. Note that $G/H \cong G^0/H^0$ and its compact dual is U/H_0. One shows that $G^C = M^+ H^{0,\mathbb{C}} M^-$ and $M^* = G^C/H^{0,\mathbb{C}} M^-$ is holomorphically diffeomorphic to the compact dual by the map $i : U/H^0 \to M^*$ given by $i(gH^0) = gH^{0,\mathbb{C}} M^-$. The symmetric space can be imbedded into its compact dual by the map $j : M = G^0/H^0 \to M^*$ defined by $j(gH^0) = g \cdot o$. This map, known as the Borel embedding, is a G^0-equivariant holomorphic diffeomorphism onto an open subset of M^*. The Harish-Chandra embedding uses this to give a realization of G^0/H^0 as a bounded domain. The image of the map $\xi : \mathfrak{m}^+ \to M^*$ defined by $\xi(x) = (\exp x) \cdot o$ is an open subset containing M. The map ξ is a $H^{0,\mathbb{C}}$-equivariant holomorphic diffeomorphism of $\mathfrak{m}^+$ onto its image and $\mathcal{D} = \xi^{-1}(M)$ is a bounded domain in $\mathfrak{m}^+$.

These theorems are trivially satisfied if we change (G^0, H^0) by $(\mathrm{Ad}\, G, \mathrm{Ad}_G H)$. Take now any finite covering $\Gamma \to G \xrightarrow{\pi} \mathrm{Ad}\, G$ and $H = \pi^{-1}(\mathrm{Ad}_G H)$ such that $G/H \cong G^0/H^0$. The action of G on $\mathrm{Ad}\, G/\mathrm{Ad}_G H$ is given by $g \cdot a \,\mathrm{Ad}_G H = \pi(g) a \,\mathrm{Ad}_G H$, and G also acts on M^* in the same way. This makes it possible to have a G-equivariant Borel embedding. Moreover, Γ acts trivially on $\mathfrak{m}^+$, as $\Gamma \subset Z(\widetilde{G})$, and it is then possible to state an $H^{\mathbb{C}}$-equivariant version of the Harish-Chandra embedding.

Theorem 2.7 (Harish-Chandra embedding)**.** *The map $\xi : \mathfrak{m}^+ \to M^*$ defined by $\xi(x) = (\exp x) \cdot o$ is an $H^{\mathbb{C}}$-equivariant holomorphic diffeomorphism of $\mathfrak{m}^+$ onto an open subset of M^* which contains M. Therefore, $\mathcal{D} = \xi^{-1}(M)$ is a bounded domain in $\mathfrak{m}^+$.*

Remark 2.8. The relation between G and G^0 may be very different from one group to another. For instance, $G^0 = \mathrm{Sp}(2n, \mathbb{R})$ in the case of $G = \mathrm{Sp}(2n, \mathbb{R})/\{\pm \mathrm{Id}\}$, $(2:1)$-covering of $\mathrm{Sp}(2n, \mathbb{R})$, the so-called metapletic group $G = \mathrm{Mp}(2n, \mathbb{R})$.

Remark 2.9. Note that given a group G with Lie algebra $\mathfrak{g}$, we can not assure the existence of a group $G^{\mathbb{C}}$ with Lie algebra $\mathfrak{g}^{\mathbb{C}}$ such that G is contained in $G^{\mathbb{C}}$. For instance, take G a non-trivial finite covering of $\mathrm{Sp}(2n, \mathbb{R})$. The group G does not sit in any group with Lie algebra $\mathfrak{sp}(2n, \mathbb{C})$, since $\mathrm{Sp}(2n, \mathbb{C})$ is simply connected and we have $\mathrm{Sp}(2n,\mathbb{R}) \subset \mathrm{Sp}(2n,\mathbb{C})$. We refer to §7 in [Vin94] for more on real forms of Lie groups.

Example 2.10. *For $G = \mathrm{SU}(1,1)$, $G^C = G^{\mathbb{C}}$, $H^{0,\mathbb{C}} = H^{\mathbb{C}}$, the decomposition $G^{\mathbb{C}} = M^+ H^{\mathbb{C}} M^-$ is given by*

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & b/d \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a - bd^{-1}c & 0 \\ 0 & d \end{pmatrix} \begin{pmatrix} 1 & 0 \\ c/d & 1 \end{pmatrix},$$

and we have that

$$\left[\begin{pmatrix} a & b \\ c & d \end{pmatrix} \right] = \left[\begin{pmatrix} 1 & b/d \\ 0 & 1 \end{pmatrix} \right] \in G^{\mathbb{C}}/H^{\mathbb{C}}M^-.$$

Therefore, the domain $\xi^{-1}(G \cdot eH^{\mathbb{C}}M^-)$ is given by

$$\left\{ \begin{pmatrix} 0 & \beta/\bar{\alpha} \\ 0 & 0 \end{pmatrix} \mid \alpha\bar{\alpha} - \beta\bar{\beta} = 1 \right\} = \left\{ \begin{pmatrix} 0 & z \\ 0 & 0 \end{pmatrix} \mid |z| < 1 \right\}.$$

We finish this section by reviewing the decomposition into restricted roots of the real algebra $\mathfrak{g}$ and its relation with the usual root decomposition of $\mathfrak{g}^{\mathbb{C}}$.

Given a decomposition $\mathfrak{g} = \mathfrak{h} + \mathfrak{m}$ from a symmetric space G/H, we define a Cartan subalgebra of $(\mathfrak{g}, \mathfrak{h})$ as a maximal subalgebra $\mathfrak{a}$ contained in $\mathfrak{m}$. It is abelian, as $[\mathfrak{m}, \mathfrak{m}] \subset \mathfrak{h}$. The dimension of such a subalgebra $\mathfrak{a}$ is called the **rank** of the symmetric space G/H, $r = \mathrm{rk}(G/H)$, and corresponds to the maximal dimension of a flat totally geodesic submanifold. Since the endomorphisms $\{\mathrm{ad}(A)\}_{A\in\mathfrak{a}}$ commute on $\mathfrak{g}$ and are symmetric with respect to B_θ we have a simultaneous diagonalization:

$$\mathfrak{g} = \mathfrak{s} + \mathfrak{a} + \sum_{\lambda\in\Sigma} \mathfrak{g}^\lambda$$

satisfying the following properties:

1. $\mathfrak{s}$ is the centralizer of $\mathfrak{a}$ in $\mathfrak{h}$ ($\mathfrak{s} + \mathfrak{a}$ is the 0-eigenspace).

2. $\Sigma = \Sigma(\mathfrak{g}, \mathfrak{a})$ is the set of the so-called restricted roots $\lambda : \mathfrak{a} \to \mathbb{R}$. It is an abstract root system.

3. $\mathfrak{g}^\lambda = \{Y \in \mathfrak{g} \mid \mathrm{ad}(A)Y = \lambda(A)Y \text{ for all } A \in \mathfrak{a}\}$, not neccesarily 1-dimensional.

4. $[\mathfrak{g}^\lambda, \mathfrak{g}^{\lambda'}] \subset \mathfrak{g}^{\lambda+\lambda'}$.

Now we consider the usual root decomposition. Generally, a Cartan subalgebra of $\mathfrak{g}^{\mathbb{C}}$ projects both into $\mathfrak{h}^{\mathbb{C}}$ and $\mathfrak{m}^{\mathbb{C}}$, but in the Hermitian case, we obtain a Cartan subalgebra by complexifying a maximal abelian subalgebra $\mathfrak{t}$ of $\mathfrak{h}$ (cf. [Kna02], VII.9). We consider the usual root space decomposition for the root system $\Delta = \Delta(\mathfrak{g}^{\mathbb{C}}, \mathfrak{t}^{\mathbb{C}})$:

$$\mathfrak{g}^{\mathbb{C}} = \mathfrak{t}^{\mathbb{C}} + \sum_{\alpha \in \Delta} \mathfrak{g}^{\mathbb{C}}_{\alpha}$$

such that

1. $\mathfrak{g}^{\mathbb{C}}_{\alpha} = \{Y \in \mathfrak{g} \mid \operatorname{ad}(h)Y = \alpha(h)Y \;\text{ for all } h \in \mathfrak{t}^{\mathbb{C}}\}$ is 1-dimensional.
2. $[\mathfrak{g}^{\mathbb{C}}_{\alpha}, \mathfrak{g}^{\mathbb{C}}_{\beta}] = \mathfrak{g}^{\mathbb{C}}_{\alpha+\beta}$ is satisfied.

For each root $\alpha \in \Delta$ we define $h_\alpha \in \mathfrak{t}^{\mathbb{C}}$ such that

$$\alpha(h) = 2\frac{B(h, h_\alpha)}{B(h_\alpha, h_\alpha)} \text{ for every } h \in \mathfrak{t}^{\mathbb{C}}$$

and $e_\alpha \in \mathfrak{g}^{\mathbb{C}}_{\alpha}$ such that $\tau e_\alpha = -e_{-\alpha}$ and $[e_\alpha, e_{-\alpha}] = h_\alpha$. We have that $h_\alpha \in i\mathfrak{t}$.

Since $\operatorname{ad}(\mathfrak{t}^{\mathbb{C}})$ preserves $\mathfrak{h}^{\mathbb{C}}$ and $\mathfrak{m}^{\mathbb{C}}$, $\mathfrak{g}^{\mathbb{C}}_{\alpha}$ must lie either in $\mathfrak{h}^{\mathbb{C}}$ or in $\mathfrak{m}^{\mathbb{C}}$. If $\mathfrak{g}^{\mathbb{C}}_{\alpha} \subset \mathfrak{h}^{\mathbb{C}}$ we say that a root α is **compact** and denote the set of compact roots by Δ_C. Otherwise, $\mathfrak{g}^{\mathbb{C}}_{\alpha} \subset \mathfrak{m}^{\mathbb{C}}$ and we say that α is **non-compact**. We denote the set of non-compact roots by Δ_Q.

We choose an ordering of the roots in such a way that $\mathfrak{m}^+$ is spanned by the root vectors corresponding to the non-compact positive roots, and $\mathfrak{m}^-$ by those corresponding to the non-compact negative ones. For instance, any lexicographical ordering given by any ordered basis of $i\mathfrak{t}$ starting with the element $-iJ$. In this case, for a non-compact root α, $[J, e_\alpha] = \alpha(J)e_\alpha$, with $\alpha(J) = i$ when $\alpha(-iJ) > 0$ (positive root), and $\alpha(J) = -i$ when $\alpha(-iJ) < 0$ (negative root). In addition, every positive non-compact root is larger than any compact root.

We use the superscript + (resp. -) to denote the positive (resp. negative) roots from a set of roots: Δ^+, Δ^+_C, Δ^+_Q (resp. Δ^-, Δ^-_C, Δ^-_Q). We have then that

$$\mathfrak{m}^{\pm} = \sum_{\alpha \in \Delta^+_Q} \mathbb{C}e_{\pm\alpha}.$$

Remark 2.11. With the choice of this ordering we are following Drucker's notation ([Dru78]), instead of Helgason's, where the positive roots α satisfy $\alpha(J) = -i$ ([Hel01], Ch. VIII, 7.13) .

We define a real basis of $\mathfrak{m}$ by:

$$x_\alpha = e_\alpha + e_{-\alpha} \qquad\qquad y_\alpha = i(e_\alpha - e_{-\alpha})$$

Two roots $\alpha, \beta \in \Delta$ are said to be **strongly orthogonal** if neither $\alpha+\beta$ nor $\alpha-\beta$ is a root (equivalently $[\mathfrak{g}^\alpha, \mathfrak{g}^{\pm\beta}] = \{0\}$). We construct a maximal set of strongly orthogonal non-compact roots as follows. Let γ_1 be the lowest root in Δ_Q^+ with respect to the ordering defined above. Consider $T_2 = \{\alpha \in \Delta_Q^+ \mid \alpha \text{ is strongly orthogonal to } \gamma_1\}$ and choose the lowest root γ_2 in T_2. Repeat this process: from γ_j, the lowest root in T_j, we define γ_{j+1} as the lowest root of $T_{j+1} = \{\alpha \in T_j \mid \alpha \text{ is strongly orthogonal to } \gamma_j\}$. This process can be repeated $r = \operatorname{rk}(G/H)$ times. For abbreviation, such a set $\Gamma := T_r$ is called a **system of st-orthogonal roots**.

Lemma 2.12. *For a system of st-orthogonal roots $\Gamma = \{\gamma_1, \dots, \gamma_r\}$, define:*

$$\mathfrak{a} = \sum_{\gamma\in\Gamma} \mathbb{R}x_\gamma \qquad \mathfrak{a}^+ = \sum_{\gamma\in\Gamma} \mathbb{R}e_\gamma \qquad \mathfrak{a}^- = \sum_{\gamma\in\Gamma} \mathbb{R}e_{-\gamma}.$$

The subspace $\mathfrak{a}$ is a Cartan subalgebra of the pair $(\mathfrak{g}, \mathfrak{h})$, and $\mathfrak{a}^\pm$ are the images of $\mathfrak{a}$ by the isomorphisms in Lemma 2.6. Furthermore, the group H acts transitively by Ad *on all Cartan subalgebras of $(\mathfrak{g}, \mathfrak{h})$, and $\mathfrak{m} = \operatorname{Ad}(H)\mathfrak{a}$.*

Lemma 2.13. *Any Cartan subalgebra of $(\mathfrak{g}, \mathfrak{h})$ is of the form $\sum_{\gamma\in\Gamma} \mathbb{R}x_\gamma$ for some system of st-orthogonal roots Γ.*

Proof. Since the group H acts transitively on the Cartan subalgebras, it suffices to study $\operatorname{Ad}(h)\mathfrak{a}$, and more concretely $\operatorname{Ad}(h)x_\gamma$ for $h \in H$. Recall that $x_\gamma = e_\gamma + e_{-\gamma}$. Given any root γ, we have that $\operatorname{ad}(X)e_\gamma = \gamma(X)e_\gamma$ for $X \in \mathfrak{t}^\mathbb{C}$. By the Ad-equivariance of ad we have that for $X \in \mathfrak{t}^\mathbb{C}$

$$\begin{aligned}\operatorname{ad}(\operatorname{Ad}(h)X)(\operatorname{Ad}(h)e_\gamma) &= \operatorname{Ad}(h)(\operatorname{ad}(X)(e_\gamma)) = \operatorname{Ad}(h)(\gamma(X)e_\gamma)\\ &= \gamma(X)\operatorname{Ad}(h)e_\gamma = (\gamma\circ\operatorname{Ad}(h^{-1}))(\operatorname{Ad}(h)X)\operatorname{Ad}(h)e_\gamma,\end{aligned}$$

i.e., $\operatorname{Ad}(h)e_\gamma$ is a root vector for the root $\gamma\circ\operatorname{Ad}(h^{-1}) \in \Delta(\mathfrak{g}^\mathbb{C}, \operatorname{Ad}(h)\mathfrak{t}^\mathbb{C})$. Also by Ad-equivariance, the action of $\operatorname{Ad}(h)$ preserves strong orthogonality, and therefore $\Gamma\circ\operatorname{Ad}(h^{-1})$ is a system of st-orthogonal roots and we have that

$$\operatorname{Ad}(h)\mathfrak{a} = \sum_{\gamma\in\Gamma\circ\operatorname{Ad}(h^{-1})} \mathbb{R}x_\gamma.$$

Therefore, every Cartan subalgebra has the claimed form. $\square$

2.2 Cayley transform

For a st-orthogonal system of roots Γ, consider

$$x_\Gamma = \sum_{\gamma\in\Gamma} x_\gamma, \qquad y_\Gamma = \sum_{\gamma\in\Gamma} y_\gamma, \qquad e_\Gamma = \sum_{\gamma\in\Gamma} e_\gamma,$$

$$c_\gamma = \exp\left(\frac{\pi}{4} i y_\gamma\right) \in U \text{ for } \gamma \in \Gamma, \qquad c = \prod_{\gamma\in\Gamma} c_\gamma = \exp\left(\frac{\pi}{4} i y_\Gamma\right) \in U \subset G^C.$$

We define the **Cayley transform** as the action of the element c:

- on the Lie algebra $\mathfrak{g}^{\mathbb{C}}$ by $\mathrm{Ad}(c) : \mathfrak{g}^{\mathbb{C}} \to \mathfrak{g}^{\mathbb{C}}$,
- on the domain $\mathcal{D} \subset \mathfrak{m}^+$ by $\xi^{-1}(c\xi(\mathcal{D}))$.

Remark 2.14. If we start with G/H, the element c lies in G^C, which is not necessarily the complexification of G and may not even contain it. Nonetheless, the action on the Lie algebra $\mathfrak{g}^{\mathbb{C}}$ and on the domain is well defined.

Example 2.15. *Every $\mathfrak{g}[\gamma] = \langle e_\gamma, e_{-\gamma}, h_\gamma\rangle$ is isomorphic to $\mathfrak{sl}(2,\mathbb{C})$ via the identification $e_\gamma = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $e_{-\gamma} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ and $h_\gamma = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. Then, regarding that $y_\gamma = i(e_\gamma - e_{-\gamma})$, we have that $c_\gamma = \exp \frac{\pi}{4} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$.*

Example 2.16. *We consider the real group $G = \mathrm{SU}(1,1)$. The action of $c = c_\gamma$ on $\mathfrak{sl}(2,\mathbb{C})$ is given by*

$$Ad(c_\gamma) : \begin{cases} h_\gamma \mapsto x_\gamma \\ x_\gamma \mapsto -h_\gamma \\ y_\gamma \mapsto y_\gamma. \end{cases}$$

And for the action on the domain we have

$$c\begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & z+i \\ i & zi+1 \end{pmatrix} \in c\mathcal{D}, \quad \left[\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & z+i \\ 1 & zi+1 \end{pmatrix}\right] = \left[\begin{pmatrix} 1 & \frac{z+i}{zi+1} \\ 0 & 1 \end{pmatrix}\right],$$

$$\xi^{-1}(c\mathcal{D}) = \left\{\begin{pmatrix} 0 & i\frac{1-iz}{1+iz} \\ 0 & 0 \end{pmatrix} \mid |z| < 1\right\} = \left\{\begin{pmatrix} 0 & z \\ 0 & 0 \end{pmatrix} \mid \Re z > 0\right\} \cong \mathcal{H},$$

giving the usual diffeomorphism between the Poincaré disk and the upper half-plane $\mathcal{H}$, known as the Cayley transform.

In this case we also have an action on the group, $c\mathrm{SU}(1,1)c^{-1} = \mathrm{SL}(2,\mathbb{R}) \subset \mathrm{SL}(2,\mathbb{C})$. The group $\mathrm{SL}(2,\mathbb{R})$ is not the isometry group of $\mathcal{H}$, but a $(2:1)$-cover.

We first study the action of the Cayley transform on the Lie algebra $\mathfrak{g}^{\mathbb{C}}$. One shows that the Cayley transform $\mathrm{Ad}(c)$ satisfies $\mathrm{Ad}(c^8) = \mathrm{Id}$, $\mathrm{Ad}(c) \circ \theta = \theta \circ \mathrm{Ad}(c^{-1})$, and consequently $\mathrm{Ad}(c^4)$ preserves $\mathfrak{h}$ and $\mathfrak{m}$, even though $\mathrm{Ad}(c)$ does not preserve $\mathfrak{g}$. As $(\mathrm{Ad}(c^4))^2 = \mathrm{Id}$, we decompose $\mathfrak{h}$ and $\mathfrak{m}$ into ± 1-eigenspaces for $\mathrm{Ad}(c^4)$:

$$\mathfrak{m} = \mathfrak{m}_T + \mathfrak{m}_2 \qquad\qquad \mathfrak{h} = \widetilde{\mathfrak{h}}_T + \mathfrak{q}_2.$$

We define $\widetilde{\mathfrak{g}}_T = \widetilde{\mathfrak{h}}_T + \mathfrak{m}_T$, which is a Lie algebra as $\widetilde{\mathfrak{h}}_T$ acts on $\mathfrak{m}_T$. Since $\widetilde{\mathfrak{h}}_T$ may have a non-trivial ideal, we define

$$\mathfrak{h}_T = [\mathfrak{m}_T, \mathfrak{m}_T] \qquad\qquad \mathfrak{g}_T = \mathfrak{h}_T + \mathfrak{m}_T.$$

to get an irreducible Hermitian symmetric pair $(\mathfrak{g}_T, \mathfrak{h}_T)$.

The subalgebras $\widetilde{\mathfrak{g}}_T$ and $\widetilde{\mathfrak{h}}_T$ are then the normalizers

$$\widetilde{\mathfrak{g}}_T = \mathfrak{n}_{\mathfrak{g}}(\mathfrak{g}_T) \qquad\qquad \widetilde{\mathfrak{h}}_T = \mathfrak{n}_{\mathfrak{h}}(\mathfrak{h}_T).$$

We also use the following notation

$$\mathfrak{m}_T^{\pm} = \mathfrak{m}_T^{\mathbb{C}} \cap \mathfrak{m}^{\pm} \qquad\qquad \mathfrak{m}_2^{\pm} = \mathfrak{m}_2^{\mathbb{C}} \cap \mathfrak{m}^{\pm}.$$

We denote the corresponding analytic subgroups of G for $\mathfrak{g}_T$ and $\mathfrak{h}_T$ by $G_T \subset G$ and $H_T \subset H$, which is a maximal compact subgroup. We have that $G_T = H_T \exp \mathfrak{m}_T$. The corresponding analytic subgroups of $\widetilde{\mathfrak{g}}_T$ and $\widetilde{\mathfrak{h}}_T$ are $N_G(\mathfrak{g}_T)_0$ and $N_H(\mathfrak{h}_T)_0$. The Cartan decomposition is $\mathfrak{n}_{\mathfrak{g}}(\mathfrak{g}_T) = \mathfrak{n}_{\mathfrak{h}}(\mathfrak{h}_T) + \mathfrak{m}_T$ and the maximal compact subgroup of $N_G(\mathfrak{g}_T)_0$ is $N_H(\mathfrak{h}_T)_0$, whose complexification is $N_{H^{\mathbb{C}}}(\mathfrak{h}_T^{\mathbb{C}})_0$.

Example 2.17. *For $G = SU(1,2)$, take the Cartan subalgebra $\mathfrak{t}^{\mathbb{C}} \subset \mathfrak{g}^{\mathbb{C}}$ consisting of diagonal matrices $\{diag(x_1, x_2, x_3) \mid x_1 + x_2 + x_3 = 0\}$. Let $e_j(diag(x_1, x_2, x_3)) = x_j$. The root system is $\{e_i - e_j\}_{1 \leq i \neq j \leq 3}$, the rank is 1 and we can take $\Gamma = \{e_2 - e_1\}$. We have*

$$y_\Gamma = \left(\begin{array}{c|cc} & 0 & 0 \\ \hline 1 & & \\ 0 & & \end{array}\right) \qquad c_\Gamma = \frac{1}{\sqrt{2}} \left(\begin{array}{c|cc} 1 & -1 & 0 \\ \hline 1 & 1 & 0 \\ 0 & 0 & \sqrt{2} \end{array}\right) \qquad c_\Gamma^4 = \left(\begin{array}{c|cc} -1 & & \\ \hline & -1 & \\ & & 1 \end{array}\right)$$

The unit matrix E_{ij} is a root vector for the root $e_i - e_j$, so we can use any entry of the matrix to represent a root space or part of the Cartan subalgebra. Using this convention, the ± 1-eigenspaces of the action $\mathrm{Ad}(c^4)$ on $\mathfrak{g}^{\mathbb{C}}$ are represented by $\left(\begin{array}{c|cc} + & + & - \\ \hline + & + & - \\ - & - & + \end{array}\right)$, and we then have

$$\widetilde{\mathfrak{h}}_T = \left(\begin{array}{c|cc} * & & \\ \hline & * & 0 \\ & 0 & * \end{array}\right) \qquad \mathfrak{m}_T = \left(\begin{array}{c|cc} & * & 0 \\ \hline * & & \\ 0 & & \end{array}\right) \qquad \mathfrak{h}_T = \left(\begin{array}{c|cc} * & & \\ \hline & * & 0 \\ & 0 & 0 \end{array}\right).$$

As $\mathrm{Ad}(c^4) = \mathrm{Id}$ on $\mathfrak{g}_T$, we have that $(\mathrm{Ad}(c^2))^2 = \mathrm{Id}$. Moreover, $\mathrm{Ad}(c^2)$ commutes with θ, so it preserves $\mathfrak{h}_T$ and we have the decomposition:

$$\mathfrak{h}_T = \mathfrak{h}' + i\mathfrak{m}', \qquad \text{into } \pm 1\text{-eigenspaces for } \mathrm{Ad}(c^2).$$

Let H' be the isotropy group of ie_Γ in H. One proves that the Lie algebra of H' and its identitity component H'_0 is $\mathfrak{h}'$.

Lemma 2.18. *The map $\psi : \mathfrak{m}'^{\mathbb{C}} \to \mathfrak{m}^+$ given by $\psi(X) = \frac{1}{2}\,\mathrm{ad}(e_\Gamma)(X)$ is an $\mathrm{Ad}(H'_0)$-equivariant complex vector space isomorphism such that $\psi(i\mathfrak{m}') = \mathfrak{n}^+$. Moreover, $[e_\Gamma, \mathfrak{h}'^{\mathbb{C}}] = 0$, and this isomorphism is not only H'_0 but $H'^{\mathbb{C}}$-equivariant.*

Lemma 2.19. *The following diagram commutes.*

$$\begin{tikzcd}
\mathfrak{h}'^{\mathbb{C}} \arrow[r, "\mathrm{ad}"] \arrow[d, "i\cdot"] & \mathrm{End}(\mathfrak{m}'^{\mathbb{C}}) \arrow[d, "\mathrm{ad}(\mathrm{ad}(e_\Gamma))"] \\
\mathfrak{h}^{\mathbb{C}} \arrow[r, "\mathrm{ad}"] & \mathrm{End}(\mathfrak{m}^+)
\end{tikzcd}$$

Proof. Given $X \in \mathfrak{h}'^{\mathbb{C}}$, for all $Y \in \mathfrak{m}^-$ we have to prove that

$$\mathrm{ad}_{\mathfrak{m}'^{\mathbb{C}}}(X)(\mathrm{ad}(e_\Gamma), Y) = \mathrm{ad}(e_\Gamma)(\mathrm{ad}_{\mathfrak{m}^+}(X)Y).$$

This is equivalent to $[X, [e_\Gamma, Y]] = [e_\Gamma, [X, Y]]$, which is true by Lemma 2.18: $[Y, [X, e_\Gamma]] = 0$ since $[\mathfrak{h}'^{\mathbb{C}}, e_\Gamma] = 0$. $\square$

Consider the bounded domain $\mathcal{D} \subset \mathfrak{m}^+$ given by the Harish-Chandra embedding (Theorem 2.7). Via $\xi : \mathfrak{m}^+ \to M^*$, the element ie_Γ corresponds to $c \cdot o$.

We recall now the main definitions about symmetric cones.

Definition 2.20. *Let $(V, \langle\, ,\, \rangle)$ be a Euclidean real vector space. A subset $\Omega \subset V$ is said to be a cone if given $x \in \Omega$ and $\lambda > 0$, $\lambda x \in \Omega$. The group of automorphisms of a cone is defined as*

$$G(\Omega) = \{g \in \mathrm{GL}(V) \mid g\Omega = \Omega\}.$$

*A cone is said to be homogeneous if $G(\Omega)$ acts transitively in Ω, and self-dual if it coincides with its dual $\{v \in V \mid \langle x, v\rangle > 0\ \forall x \in \Omega\}$. A homogeneous and self-dual cone is called **symmetric** and it is indeed a symmetric space.*

A tube domain over the cone Ω is a domain of the form

$$T_\Omega = \{u + iv \in V^{\mathbb{C}}, u \in V, v \in \Omega\}.$$

A domain $\mathcal{D}$ is said to be of **tube type** if it is biholomorphic to a tube domain T_Ω. In the case of a symmetric domain, the cone Ω is also symmetric.

The **Shilov boundary** of a bounded domain $\mathcal{D}$ is defined as the smallest closed subset $\check{S}$ of the topological boundary $\partial\mathcal{D}$ for which every function f continuous on $\overline{\mathcal{D}}$ and holomorphic on $\mathcal{D}$ satisfies that

$$|f(z)| \leq \max_{w\in\check{S}} |f(w)| \text{ for every } z \in \mathcal{D}.$$

Example 2.21. *By repeated use of the mean value theorem for one complex variable, it is proved that the Shilov boundary of the polydisc* $P = \{\sum_1^r z_j e_j \mid |z_j| < 1\}$ *is the torus* $T = \{\sum_1^r z_j e_j \mid |z_j| = 1\}$.

The Shilov boundary of a bounded domain $\mathcal{D} \subset \mathfrak{m}^+$ is the H-orbit $H \cdot e_\Gamma = H \cdot ie_\Gamma$, or alternatively the G-orbit $G \cdot e_\Gamma$. It is indeed the unique closed G-orbit in $\partial\mathcal{D}$. The space $S_T = H_T \cdot ie_\Gamma = H_T/H'_T$ is a (real) symmetric space. In the tube type case, this is the whole Shilov boundary $\check{S}$. In the non-tube type case, $\check{S}$ is a fibered space with fibre S_T and base space H/H_T (which turns out to be a Hermitian symmetric subspace of U/H).

Proposition 2.22. *Let M_0 be a non-compact Hermitian symmetric space and let $\mathcal{D}$ be its Harish-Chandra realization as a bounded symmetric domain. The following are equivalent:*

(i) $\mathcal{D}$ and M_0 are of tube type.

(ii) $\dim_\mathbb{R} \check{S} = \dim_\mathbb{C} \mathcal{D}$.

(iii) $\check{S}$ is a symmetric space of compact type.

(iv) $c^4 = \mathrm{Id}$.

(v) $\mathfrak{g}=\mathfrak{g}_T$.

In the non-tube type case, $\mathfrak{g}_T$ is a θ-invariant subalgebra of $\mathfrak{g}$. So the orbit $G_T \cdot o \subset M^*$ is a Hermitian symmetric space, a maximal isometrically embedded space of tube type. As a homogeneous space, it is G_T/H_T, and as a bounded domain it is $\mathcal{D}_T = \mathcal{D} \cap \mathfrak{m}_T^+$. It is called the maximal tube type subspace/subdomain.

Lemma 2.23. *Define* $\mathfrak{n}_T^\pm = \mathrm{Ad}(c)\mathfrak{g} \cap \mathfrak{m}_T^\pm$. *The vector space* $\mathfrak{n}_T^\pm$ *is a real form of* $\mathfrak{m}_T^\pm$. *It becomes a Euclidean vector space under the restriction of the Hermitian form B_τ, defined by $B_\tau(X,Y) = B(X,\tau Y)$, where τ is the involution fixing the compact real form of $\mathfrak{g}^\mathbb{C}$. It holds that $\Omega = H_T^* \cdot e_\Gamma$ is a homogeneous self-dual cone in $\mathfrak{n}_T^+$, and the isotropy group of e_Γ in H_T^* is H'_0, i.e., $\Omega \cong H_T^*/H'_0$*

The Cayley transform ${}^cD = c{\cdot}D \subset \mathfrak{m}^+$ for a space of tube-type gives the realization as a tube-type domain

$$^c\mathcal{D} = \{x + iy \mid x \in \mathfrak{n}_T^+, y \in \Omega\}.$$

For a space of non-tube type, consider $\Phi(u,v) = \frac{1}{2}\operatorname{ad}(u)\operatorname{ad}(v)^* e_\Gamma$, a Hermitian bilinear form $\mathfrak{m}_2^+ \times \mathfrak{m}_2^+ \to \mathfrak{m}_T^+$. The Cayley transform gives a "generalized half-plane" which is a Siegel domain of type II:

$$^c\mathcal{D} = \{x + iy + z_2 \mid x \in \mathfrak{n}_T^+, z_2 \in \mathfrak{m}_2^+, y - \Phi(z_2, z_2) \in \Omega\}.$$

Remark 2.24. Note that for tube-type domains the Shilov boundary is the compact dual of the cone Ω.

Irreducible symmetric spaces were originally classified by Cartan in [Car26] and [Car27]. The non-compact Hermitian ones correspond to those non-compact $\mathfrak{g}$ such that the centre of $\mathfrak{h}$ is one dimensional (a more straightforward classification based on the largest root can be found in [Wol64]). There are four classical families and two exceptional ones, which we mention in the table below. Furthermore, Proposition 2.22 tells which of them are of tube type.

$\mathfrak{g}$	tube type
$\mathfrak{su}(p,q)$	$p=q$
$\mathfrak{so}^*(2n)$	n even
$\mathfrak{so}(2,n)$	yes
$\mathfrak{sp}(2n,\mathbb{R})$	yes
$\mathfrak{e}_6^{-14}$	no
$\mathfrak{e}_7^{-25}$	yes

In the exceptional Lie algebras, the superindex refers to the signature of the Killing form (see, e.g., [Hel01]). In Tables C.4 and C.5, classical and exceptional groups G have been taken such that G/H is an irreducible Hermitian symmetric spaces. Table C.4 indicates the Shilov boundary $\check{S} = H/H'$, its non-compact dual the cone $\Omega = H^*/H_0'$, the isotropy representation space $\mathfrak{m}'$ and its complexification $\mathfrak{m}'^{\mathbb{C}}$, corresponding to the Cartan decomposition of the Lie algebra $\mathfrak{h}^* = \mathfrak{h}' + \mathfrak{m}'$ of H^*. Table C.5 gives the maximal symmetric space of tube type isometrically embedded in the two series of classical irreducible symmetric spaces of non-tube type. We describe also the Shilov boundaries of G/H and $\widetilde{G}/\widetilde{H}$ which are of the form $\check{S} = H/H'$, and $\widetilde{\check{S}} = \widetilde{H}/\widetilde{H}'$, respectively. Notice that in the non-tube case the Shilov boundary $\check{S}$ is a homogeneous space H/H' but it is not symmetric.

2.3 Restricted root theory

In this section we combine the Cayley transform with root theory.

2.3.1 Classical results

Let r be the rank of G/H and $\Gamma = \{\gamma_1, \dots, \gamma_r\}$ denote a system of st-orthogonal roots. Recall that given a root γ we have a three dimensional subalgebra $\langle e_\gamma, e_{-\gamma}, h_\gamma \rangle$. Let $\mathfrak{t}^- = \sum_\Gamma \mathbb{R} i h_\gamma$, and $\mathfrak{t}^+$ be its orthogonal complement with respect to $B_\tau(X,Y) = B(X, \tau Y)$ in $\mathfrak{t}$ (recall that $h_\gamma \in i\mathfrak{t}$). From the action of $\mathrm{Ad}(c_\gamma)$ and strong orthogonality, we have that the Cayley transform $\mathrm{Ad}(c)$ acts trivially on $\mathfrak{t}^+$ and interchanges $i\mathfrak{t}^-$ and $\mathfrak{a}$. Let $\mathfrak{t}' = \mathfrak{t}^+ + \mathfrak{a}$. As $\mathfrak{t}'^{\mathbb{C}} = \mathrm{Ad}(c)\mathfrak{t}^{\mathbb{C}}$, $\mathfrak{t}'^{\mathbb{C}}$ is a Cartan subalgebra of $\mathfrak{g}^{\mathbb{C}}$, and $\mathrm{Ad}(c)^{-1}$ sends the root system $\Delta(\mathfrak{g}^{\mathbb{C}}, \mathfrak{t}'^{\mathbb{C}})$ to $\Delta(\mathfrak{g}^{\mathbb{C}}, \mathfrak{t}^{\mathbb{C}})$. Similarly, the restrictions to $\mathfrak{a}$ of the $\mathfrak{t}'^{\mathbb{C}}$-roots are sent to restrictions to $i\mathfrak{t}^-$ of the $\mathfrak{t}^{\mathbb{C}}$-roots.

We study the restricted root system $\Sigma(\mathfrak{g}, \mathfrak{a})$ from the restriction to $i\mathfrak{t}^-$ of the root system $\Delta = \Delta(\mathfrak{g}^{\mathbb{C}}, \mathfrak{t}^{\mathbb{C}})$, i.e., the root system $\Sigma(\mathrm{Ad}(c)\mathfrak{g}, i\mathfrak{t}^-)$. We thus take advantage from the division into compact and non-compact roots. The theory of restricted roots in this context was originally introduced in [HC56], but we refer and use the notation of [Hel08].

We denote by $\pi : (\mathfrak{t}^{\mathbb{C}})^* \to (i\mathfrak{t}^-)^*$ the restriction to $i\mathfrak{t}^-$. When talking about restricted roots we identify γ_i with $\pi(\gamma_i)$. For instance, if we say that the restriction of α is $\frac{1}{2}\gamma_i$, we mean that $\pi(\alpha) = \frac{1}{2}\pi(\gamma_i)$.

Lemma 2.25. *([Hel08], V.3) The restriction to $i\mathfrak{t}^-$ of any compact positive root α is either 0, or $-\frac{1}{2}\gamma_i$, or $\frac{1}{2}(\gamma_j - \gamma_i)$, with $\gamma_i, \gamma_j \in \Gamma$ and $j > i$.*

We define the disjoint sets of positive compact roots:

- $C_0 = \{\alpha \in \Delta_C^+ \mid \pi(\alpha) = 0\}$,
- $C_i = \{\alpha \in \Delta_C^+ \mid \pi(\alpha) = -\frac{1}{2}\gamma_i\}$,
- $C_{ij} = \{\alpha \in \Delta_C^+ \mid \pi(\alpha) = \frac{1}{2}(\gamma_j - \gamma_i)\}$, for $j > i$.

Recall that an α-string of roots is a set of the form $\{\beta + n\alpha\}_{n\in\mathbb{Z}} \cap \Delta$ for some $\beta \in \Delta$. The α-string may consist of β by itself.

Lemma 2.26. *Let $\alpha, \beta \in \Delta$. The α-string containing β has no gaps, i.e., it is of the form $\{\beta + n\alpha \mid -p \le n \le q\}$, and p and q satisfy*

$$p - q = \frac{2\langle \alpha, \beta \rangle}{\langle \alpha, \alpha \rangle}. \tag{2.26.1}$$

We study the restriction of the non-compact roots. For $\alpha \in C_i, C_{ij}$, $\langle \alpha, \gamma_i \rangle = -\frac{1}{2}\langle \gamma_i, \gamma_i \rangle$. Thus, from Lemma 2.26, $\alpha + \gamma_i$ is a root, which is non-compact and we have that the positive non-compact roots are the union of the following disjoint sets

- $\Gamma = \{\gamma_1, \ldots, \gamma_r\}$,
- $Q_i = \{\alpha \in \Delta_Q^+ \mid \pi(\alpha) = \frac{1}{2}\gamma_i\}$,
- $Q_{ij} = \{\alpha \in \Delta_Q^+ \mid \pi(\alpha) = \frac{1}{2}(\gamma_j + \gamma_i)\}$.

The following translations are bijections

$$\begin{aligned} C_i &\xrightarrow{+\gamma_i} Q_i & \qquad Q_i &\xrightarrow{-\gamma_i} C_i \\ Q_{ij} &\xrightarrow{+\gamma_i} C_{ij} & \qquad C_{ij} &\xrightarrow{-\gamma_i} Q_{ij}. \end{aligned} \tag{2.26.2}$$

The following theorem will be fundamental in what follows.

Theorem 2.27. *The restricted root system $\Sigma(\mathrm{Ad}(c)\mathfrak{g}, i\mathfrak{t}^-)$ consists of the roots $\pm\gamma_j$ $(1 \leq j \leq r)$ with multiplicity 1, the roots $\pm\frac{1}{2}\gamma_j \pm \frac{1}{2}\gamma_k$ $(j \neq k)$ with multiplicity a, and possibly the roots $\pm\frac{1}{2}\gamma_j$ with even multiplicity $2b$. The restricted root system is then of type $(BC)_r$ or C_r (see [Hel01], Ch.X) and its Weyl group consists of the signed permutations of the set Γ.*

When $b = 0$ (resp. $b \neq 0$) the space G/H, the Lie algebra $\mathfrak{g}$ or the group G is of tube type (resp. of non-tube type). We have seen above that the spaces of tube type have a realization as a domain of tube type. The condition $b = 0$ is equivalent to $(\mathfrak{g}, \mathfrak{h})$ having a restricted root system of type C_r, which is in turn equivalent to the condition in Proposition 2.22.

We define the positive compact/non-compact roots of tube/non-tube type as

$$\Delta^+_{C,nt} = C_0 \cup \bigcup_{1 \leq j \leq r} C_j \qquad \Delta^+_{Q,nt} = \bigcup_{1 \leq j \leq r} Q_j \qquad \Delta^+_{nt} = \Delta^+_{C,nt} \cup \Delta^+_{Q,nt}$$

$$\Delta^+_{C,t} = \bigcup_{1 \leq i < j \leq r} C_{ij} \qquad \Delta^+_{Q,t} = \Gamma \cup \bigcup_{1 \leq i \neq j \leq r} Q_{ij} \qquad \Delta^+_{t} = \Delta^+_{C,t} \cup \Delta^+_{Q,t}.$$

A relevant number in what follows will be the **dual Coxeter number** N, which is defined as

$$N = a(r-1) + b + 2. \tag{2.27.1}$$

Denoting $n = \dim_{\mathbb{C}} \mathfrak{m}^+$ and $n_T = \dim_{\mathbb{C}} \mathfrak{m}^+_T$, we have

$$n_T = \frac{r(r-1)}{2}a + r \quad \text{and} \quad n = n_T + rb.$$

And therefore,

$$N = \frac{2n_T + (n - n_T)}{r} = \frac{n + n_T}{r}.$$

2.3.2 More results on restricted roots

We show now some results not found in the literature which will be needed later. The squared norm of any of the elements of Γ is an important constant that we denote by $\langle\gamma_i,\gamma_i\rangle$. It does not depend on i as all γ_i have the same length (cf. [Hel08], Ch. V, Th. 4.1., but it can be deduced also from the next lemma).

Lemma 2.28. *In the non-tube type case ($b \neq 0$), we have the following:*

1. *For any $\alpha \in Q_i, Q_{ij}, C_i, C_{ij}$, $\langle\alpha,\alpha\rangle = \langle\gamma_i,\gamma_i\rangle$, for any $1 \leq i \neq j \leq r$.*
2. *Let $\alpha \in Q_i$ and $\beta \in Q_j$. If $\beta - \alpha$ is a root, then $\langle\beta,\alpha\rangle = \frac{1}{2}\langle\gamma_i,\gamma_i\rangle$.*
3. *Let $\beta \in Q_i, Q_{ij}$. For any root $\alpha \neq \beta$, the string $\beta + n\alpha$ contains either one or two elements.*

If $b = 0$ and all the roots have the same length, the results (3) *is still true for $\alpha, \beta \in Q_{ij}$.*

Proof. We prove in detail (1), as we will repeat this type of arguments in this section. Let $\alpha \in Q_i$. The γ_i-string based on α is $\{\alpha - \gamma_i, \alpha\}$, since if $\alpha - 2\gamma_i$ and $\alpha + \gamma_i$ were roots, they would project to $-\frac{3}{2}\gamma_i$ and $\frac{3}{2}\gamma_i$ respectively, which is not possible. The α-string based on γ_i is $\{\gamma_i - \alpha, \gamma_i\}$, since if $\gamma_i - 2\alpha$ and $\gamma_i + \alpha$ were roots, they would project to 0 and $\frac{3}{2}\gamma_i$. At first glance, $\gamma_i - 2\alpha$ could be a root, but $\gamma_i - 2\alpha$ would be non-compact and only compact roots project to 0. The same argument works for $\alpha \in Q_{ij}$, but the observation about $\gamma_i - 2\alpha$ is not needed. Once we have that the strings are $\{\alpha - \gamma_i, \alpha\}$ and $\{\gamma_i - \alpha, \gamma_i\}$, we apply Lemma 2.26, and get $2\langle\gamma_i,\alpha\rangle = \langle\alpha,\alpha\rangle$ and $2\langle\alpha,\gamma_i\rangle = \langle\gamma_i,\gamma_i\rangle$, i.e., $\langle\alpha,\alpha\rangle = \langle\gamma_i,\gamma_i\rangle$. For $\alpha \in C_i, C_{ij}$, we write α as $\alpha' + \gamma_i$ for $\alpha' \in Q_i, Q_{ij}$, and $\langle\alpha,\alpha\rangle = \langle\alpha'+\gamma_i,\alpha'+\gamma_i\rangle = \langle\alpha',\alpha'\rangle + 2\langle\alpha,\gamma_i\rangle + \langle\gamma_i,\gamma_i\rangle = \langle\alpha',\alpha'\rangle$, which equals $\frac{1}{2}\langle\gamma_i,\gamma_i\rangle$ by the just proved.

For the second statement, if $\beta - \alpha$ is a root, $\beta - \alpha \in Q_{ji}$, and then $\langle\beta - \alpha, \beta - \alpha\rangle = \langle\gamma_i,\gamma_i\rangle$. From the first statement we obtain $\langle\beta,\alpha\rangle = \frac{1}{2}\langle\gamma_i,\gamma_i\rangle$.

The only case in which a string could have more than two elements is that of α, β in the same set Q_i or Q_{ij}, in such a way that $\{\beta - 2\alpha, \beta - \alpha, \beta\}$ projects to $\{-\frac{1}{2}\gamma_i, 0, \frac{1}{2}\gamma_i\}$ or $\{-\frac{1}{2}(\gamma_i - \gamma_j), 0, \frac{1}{2}(\gamma_i - \gamma_j)\}$. But this will imply $2\langle\alpha,\alpha\rangle = 2\langle\beta,\alpha\rangle = \langle\gamma_i,\gamma_i\rangle$, which is not possible from the first statement.

Since the arguments depend only on the length of the roots, the last statement follows. □

Remark 2.29. In the tube type case, we may have that $\langle\alpha,\alpha\rangle \neq \langle\gamma_i,\gamma_i\rangle$. Between the irreducible cases, this only happens for $\mathfrak{sp}(2n,\mathbb{R})$.

Lemma 2.30. *Let $\beta \in Q_i$. Then, for exactly half of the roots $\alpha \in Q_{ij}$, $\beta - \alpha$ is a root.*

Proof. We use the notation $\alpha \pm S = \{\alpha \pm s \mid s \in S\}$. First, we show that the sets $(\beta + Q_j) \cap \Delta$ and $[(\beta - \gamma_i) + (Q_j - \gamma_j)] \cap \Delta = [(\beta - \gamma_i) - C_j] \cap \Delta$ are disjoint. Note that if $\beta = \frac{1}{2}\gamma_i + \beta^\perp$, for $\beta^\perp \in (\mathfrak{t}^{\mathbb{C}})^*$, then, $\langle\beta^\perp,\beta^\perp\rangle = \frac{3}{4}\langle\gamma_i,\gamma_i\rangle$ by (1) in Lemma 2.28. Let $\alpha_p, \alpha_q \in Q_j$ be such that $\beta + \alpha_p = (\beta - \gamma_i) + (\alpha_q - \gamma_j)$. Write $\alpha_p = \frac{1}{2}\gamma_j + \alpha_p^\perp$ and $\alpha_q = \frac{1}{2}\gamma_j + \alpha_q^\perp$. By (2) of Lemma 2.28, $\langle\beta,\alpha_p\rangle = \langle\beta^\perp,\alpha_p^\perp\rangle = \frac{1}{2}\langle\gamma_i,\gamma_i\rangle$. From $\beta + \alpha_p = (\beta - \gamma_i) + (\alpha_q - \gamma_j)$ we have $2\beta^\perp - \alpha_p^\perp - \alpha_q^\perp = 0$. But this cannot be true as $\langle 2\beta^\perp - \alpha_p^\perp - \alpha_q^\perp, \beta^\perp\rangle = -\frac{1}{4}\langle\gamma_i,\gamma_i\rangle$. So, the sets are disjoint.

Secondly, we show that C_{ij} is contained into the union of $\beta_i + Q_j$ and $(\beta - \gamma_i) + (Q_j - \gamma_j)$. Given $\delta \in C_{ij}$, we have that $\langle\beta - \gamma_i, \delta\rangle = \langle\beta,\delta\rangle - \frac{1}{2}\langle\gamma_i,\gamma_i\rangle$. So, at least one of $\langle\beta - \gamma_i,\delta\rangle$ or $\langle\beta,\delta\rangle$ is not zero.

And finally, it remains to show that exactly one half of C_{ij} comes from the sum of non-compact roots, and the other half from the sum of compact roots. This comes from $\langle\beta,\alpha\rangle = \langle\beta - \gamma_i, \alpha - \gamma_j\rangle$, which implies that for every root $\beta - \alpha$ which is sum of non-compact roots, there is a different root $(\beta - \gamma_i) - (\alpha - \gamma_j)$ which is sum of compact ones. □

Remark 2.31. From this lemma, if $b \neq 0$, then between the b sums of $\beta_i + Q_j$ there are $\frac{a}{2}$ roots, and $b - \frac{a}{2}$ sums which are not roots. This implies that a is even and $2b \geq a$.

Lemma 2.32. *Let $\beta \in Q_i$. Then we have that*

$$\langle\beta,\alpha\rangle = \begin{cases} \langle\gamma_i,\gamma_i\rangle & \textit{for } \alpha = \beta \\ \frac{1}{2}\langle\gamma_i,\gamma_i\rangle & \textit{for } \alpha = \beta - \gamma_i \\ \frac{1}{2}\langle\gamma_i,\gamma_i\rangle & \textit{for } \alpha \in Q_i, \alpha \neq \beta \\ 0 & \textit{for } \alpha \in C_i, \alpha \neq \beta - \gamma_i. \end{cases}$$

Proof. The first case follows directly from (1) in Lemma 2.28. For $\alpha \in Q_i \setminus \{\beta\}$ we have that

$$\langle\beta,\alpha - \gamma_i\rangle = \langle\beta,\alpha\rangle - \frac{1}{2}\langle\gamma_i,\gamma_i\rangle.$$

If we do $\alpha = \beta$ we have the case $\alpha = \beta - \gamma_i$. The sum $\alpha + \beta$ is not a root, as it would be compact and project to γ_i. So, $\langle\alpha,\beta\rangle \geq 0$, and from Lemma 2.28, (2), we have that $\langle\alpha,\beta\rangle$ is 0 or $\frac{1}{2}\langle\gamma_i,\gamma_i\rangle$. From the equation, only one of $\langle\beta,\alpha\rangle$, $\langle\beta,\alpha - \gamma_i\rangle$ can be non-zero. Let us see that $\langle\beta,\alpha - \gamma_i\rangle = 0$. If $\langle\beta,\alpha - \gamma_i\rangle \neq 0$, then one of $\beta \pm (\alpha - \gamma_i)$ would be a root, but $\beta + (\alpha - \gamma_i)$ would be compact projecting to 0, and

$\beta - (\alpha - \gamma_i)$ would project to γ_i and therefore it would be γ_i, which is only possible when $\beta = \alpha$. $\square$

2.3.3 Sums of roots

We take advantage of the decomposition into compact and non-compact roots to state some results about sums of roots which will be relevant when proving facts about the Toledo character to be defined below.

For the sake of simplicity, we use the following notation for any indexed summands $f(i)$, $f(i,j)$ and any sum over a set S:

$$\sum_i f(i) := \sum_{1\le i\le r} f(i) \qquad \sum_{i<j} f(i,j) := \sum_{1\le i<j\le r} f(i,j) \qquad \sum_S \alpha := \sum_{\alpha\in S} \alpha$$

An example will clarify this:

$$\sum_{i<j}\sum_{Q_{ij}} \alpha := \sum_{1\le i<j\le r}\sum_{\alpha\in Q_{ij}} \alpha.$$

The following lemma computes the sum of all the positive non-compact roots of tube type, $\Delta^+_{Q,t} = \Gamma \cup \bigcup_{1\le i<j\le r} Q_{ij}$.

Lemma 2.33. *For any $1 \le i < j \le r$, we have that*

$$\sum_{\alpha\in Q_{ij}} \alpha = \frac{a}{2}(\gamma_i + \gamma_j).$$

As a consequence,

$$\sum_{\alpha\in\Delta^+_{Q,t}} \alpha = \frac{N-b}{2}\sum_{j=1}^{r}\gamma_j.$$

Proof. Consider the bijections (2.26.2). On the one hand, $\sum_{\alpha\in Q_{ij}} \alpha - a\gamma_i = \sum_{\alpha\in C_{ij}} \alpha$. On the other hand, $\sum_{\alpha\in Q_{ij}} \alpha - a\gamma_j = -\sum_{\alpha\in C_{ij}} \alpha$. Adding the two expressions we obtain $\sum_{\alpha\in Q_{ij}} = \frac{a}{2}(\gamma_i + \gamma_j)$, and adding for all possible i and j:

$$\sum_{\Delta^+_{Q,t}} \alpha = \sum_{i<j}\sum_{Q_{ij}} \alpha + \sum_{j=1}^{r}\gamma_j = \frac{a(r-1)+2}{2}\sum_{j=1}^{r}\gamma_j = \frac{N-b}{2}\sum_{j=1}^{r}\gamma_j,$$

where the last equality follows from the definition of the dual Coxeter number, (2.27.1). $\square$

For the positive non-compact roots of non-tube type, $\Delta^+_{Q,nt} = \bigcup_{1\le i\le r} Q_i$ we do not have a similar result for its sum. But the following lemma will suffice for our purposes.

Lemma 2.34. *For $\beta \in Q_{ij}$ we have that*

$$\left\langle \sum_{\Delta^+_{Q,nt}} \alpha, \beta \right\rangle = \left\langle \frac{b}{2} \sum_{j=1}^{r} \gamma_j, \beta \right\rangle$$

And for $\beta \in Q_i$ we have that,

$$\left\langle \sum_{\Delta^+_{Q,nt}} \alpha, \beta \right\rangle = \left\langle \frac{N+b}{2} \sum_{j=1}^{r} \gamma_j, \beta \right\rangle.$$

Proof. Consider the expression

$$P = \left\langle \sum_{i<j} \sum_{Q_{ij}} \alpha, \sum_{k} \sum_{Q_k} \beta \right\rangle.$$

On one hand, by Lemma 2.33,

$$P = \left\langle \frac{a(r-1)}{2} \sum_{j=1}^{r} \gamma_j, \sum_{k} \sum_{Q_k} \beta \right\rangle = br \cdot \frac{a(r-1)}{2} \frac{1}{2} \langle \gamma_i, \gamma_i \rangle.$$

On the other hand, for any $\alpha \in Q_{ij}$ we have

$$\left\langle \alpha, \sum_{k} \sum_{Q_k} \beta \right\rangle = \left\langle \alpha, \sum_{Q_i \cup Q_j} \beta \right\rangle = b \langle \frac{1}{2}\gamma_i, \frac{1}{2}\gamma_i \rangle + b \langle \frac{1}{2}\gamma_j, \frac{1}{2}\gamma_j \rangle = \frac{b}{2} \langle \gamma_i, \gamma_i \rangle.$$

So we have

$$P = \frac{ar(r-1)}{2} \left\langle \alpha, \sum_{k} \sum_{Q_k} \beta \right\rangle.$$

Therefore, for $\alpha \in Q_{ij}$,

$$\left\langle \sum_{k} \sum_{Q_k} \beta, \alpha \right\rangle = \frac{2P}{ar(r-1)} \frac{b}{2} \frac{1}{2} \langle \gamma_i, \gamma_i \rangle = \left\langle \frac{b}{2} \sum_{j=1}^{r} \gamma_j, \beta \right\rangle.$$

For $\beta \in Q_i$, from Lemmas 2.30 and 2.32 we have that

$$\begin{aligned} \left\langle \sum_{j} \sum_{Q_j} \alpha, \beta \right\rangle &= \left[(r-1)\frac{a}{2} + (b-1) + 2 \right] \frac{1}{2} \langle \gamma_i, \gamma_i \rangle \\ &= \frac{N+b}{2} \cdot \frac{1}{2} \langle \gamma_i, \gamma_i \rangle = \left\langle \frac{N+b}{2} \sum_{j=1}^{r} \gamma_j, \beta \right\rangle. \end{aligned}$$

□

From Lemmas 2.33 and 2.34, we obtain the following proposition.

Proposition 2.35. *For $\beta \in \Delta^+_Q$:*

$$\left\langle \sum_{\alpha \in \Delta^+_Q} \alpha, \beta \right\rangle = N \langle \gamma_i, \gamma_i \rangle.$$

2.4 The Toledo character

Let G be a connected simple Lie group of Hermitian type. Let $H \subset G$ be a maximal compact subgroup. Assume that G/H is irreducible. We denote by B the restriction of the Killing form of $\mathfrak{g}^{\mathbb{C}}$ to $\mathfrak{h}^{\mathbb{C}}$.

2.4.1 Definition of the Toledo character

In this section we define a special character for the algebra $\mathfrak{h}^{\mathbb{C}}$, which we call Toledo character, using the Cartan decomposition $\mathfrak{g} = \mathfrak{h} + \mathfrak{m}$. Recall that a character of a complex Lie algebra $\mathfrak{h}^{\mathbb{C}}$ is a complex linear map $\mathfrak{h}^{\mathbb{C}} \to \mathbb{C}$ which factors through the quotient map $\mathfrak{h}^{\mathbb{C}} \to \mathfrak{h}^{\mathbb{C}}/[\mathfrak{h}^{\mathbb{C}}, \mathfrak{h}^{\mathbb{C}}]$. Let $\mathfrak{z}$ be the centre of $\mathfrak{h}$, so as $\mathfrak{z}^{\mathbb{C}}$ is the centre of $\mathfrak{h}^{\mathbb{C}}$. Since $\mathfrak{h}^{\mathbb{C}}$ is reductive, we have that $\mathfrak{h}^{\mathbb{C}}/[\mathfrak{h}^{\mathbb{C}}, \mathfrak{h}^{\mathbb{C}}] \cong \mathfrak{z}^{\mathbb{C}}$, and the characters of $\mathfrak{h}^{\mathbb{C}}$ are in correspondence with $(\mathfrak{z}^{\mathbb{C}})^*$.

Let $\mathfrak{t} \subset \mathfrak{h}$ be a maximal abelian subalgebra. Since $(\mathfrak{g}, \mathfrak{h})$ is a Hermitian symmetric pair, as mentioned in Section 2.1, $\mathfrak{t}^{\mathbb{C}}$ is a Cartan subalgebra of $\mathfrak{g}^{\mathbb{C}}$ and the root system $\Delta(\mathfrak{g}^{\mathbb{C}}, \mathfrak{t}^{\mathbb{C}}) \subset (\mathfrak{t}^{\mathbb{C}})^*$ decomposes into compact, Δ_C, and non-compact, Δ_Q, roots. Let N be the dual Coxeter number of $\mathfrak{g}$ defined in equation (2.27.1). We consider the element χ_T of the dual of the Lie algebra $\mathfrak{t}^{\mathbb{C}}$ given by

$$\chi_T = \frac{2}{N} \sum_{\alpha \in \Delta_Q^+} \alpha,$$

where Δ_Q^+ are the positive non-compact roots.

Remark 2.36. For groups of tube type, by Lemma 2.34 we have that, for any system of st-orthogonal roots $\Gamma \subset \Delta^+(\mathfrak{g}^{\mathbb{C}}, \mathfrak{t}^{\mathbb{C}})$, $\chi_T = \sum_{\gamma \in \Gamma} \gamma$.

The dual of χ_T with respect to B is given by

$$s_{\chi_T} = \frac{2}{N} \sum_{\alpha \in \Delta_Q^+} s_\alpha,$$

and belongs to $i\mathfrak{t}$, since every $s_\alpha = \frac{2h_\alpha}{B(h_\alpha, h_\alpha)}$ does.

Lemma 2.37. *The element s_{χ_T} belongs to $i\mathfrak{z}$, and thus defines a character of $\mathfrak{h}^{\mathbb{C}}$.*

Proof. It is immediate that s_{χ_T} commutes with $\mathfrak{t}^{\mathbb{C}}$. Since $\mathfrak{h}^{\mathbb{C}} = \mathfrak{t}^{\mathbb{C}} \oplus \bigoplus_{\beta \in \Delta_C} \mathbb{C} e_\beta$, we check that s_{χ_T} commutes with e_β for $\beta \in \Delta_C$. We have that

$$\left[\sum_{\alpha \in \Delta_Q^+} s_\alpha, e_\beta \right] = \sum_{\alpha \in \Delta_Q^+} [s_\alpha, e_\beta] = \sum_{\alpha \in \Delta_Q^+} \beta(s_\alpha) e_\beta = \left(\sum_{\alpha \in \Delta_Q^+} \langle \beta, \alpha \rangle \right) e_\beta.$$

We will see now that $\sum_{\alpha\in\Delta_Q^+}\beta(s_\alpha)$ equals zero by proving that the sum of any β-string of roots over β is zero. Let $\{\alpha+n\beta\in\Delta_Q^+ :\ \text{for } -p\leq n\leq q\}$ be a β-string of roots for some fixed root α. The sum of all these roots is $(p+q+1)\alpha+\frac{(p+q+1)(q-p)}{2}\beta$. By (2.26.1),

$$p-q=\frac{2\langle\beta,\alpha\rangle}{\langle\beta,\beta\rangle},$$

which is equivalent to

$$\langle 2\alpha+(q-p)\beta,\beta\rangle=0,$$

and multiplying by $(p+q+1)/2$ we get

$$\left\langle (p+q+1)\alpha+\frac{(p+q+1)(q-p)}{2}\beta,\beta\right\rangle=0.$$

Therefore, the inner product of a sum of a β-string with β is zero. The same happens to a β-string of length 1, $\{\alpha\}$, since $\langle\alpha,\beta\rangle=0$. Note that one root can not be at two differents β-strings. This allow us to split Δ_Q^+ into disjoint β-strings, considering strings of length 1 if necessary, and we conclude that

$$\sum_{\alpha\in\Delta_Q^+}\langle\beta,\alpha\rangle=0.$$

The dual with respect to B of $s_{\chi_T}\in i\mathfrak{z}$ considered as an element of $\mathfrak{z}^{\mathbb{C}}$ is an element of $(\mathfrak{z}^{\mathbb{C}})^*$ which defines a character of $\mathfrak{h}^{\mathbb{C}}$. □

Since this character $\chi_T:\mathfrak{h}^{\mathbb{C}}\to\mathbb{C}$ will be used in Section 3.3 to define the Toledo invariant, we call it **Toledo character**. Its definition is independent of the choice of $\mathfrak{t}^{\mathbb{C}}$. Indeed, any two such Cartan subalgebras are conjugate by an element of $h\in H^{\mathbb{C}}$, and the characters χ_T and $\chi_T\circ\mathrm{Ad}(h^{-1})$ are equal since $\mathrm{Ad}(h)$ fixes $\mathfrak{z}^{\mathbb{C}}$, the centre of $\mathfrak{h}^{\mathbb{C}}$.

We study now the condition under which a rational multiple of the character χ_T lifts to the group $H^{\mathbb{C}}$. Let $J\in\mathfrak{z}$ be the element defining the complex structure on $\mathfrak{m}$. The centre of $H^{\mathbb{C}}$, $Z^{\mathbb{C}}$, can be written as $\{e^{\theta J}\}_{\theta\in\mathbb{C}}$. First, we regard χ_T as a character of $\mathfrak{z}^{\mathbb{C}}$ and study the lifting to the identity component of the centre, $Z_0^{\mathbb{C}}$, in Lemma 2.38. The order of the element $e^{2\pi J}\in Z(G)$, which belongs to $Z(H)\subset Z(H^{\mathbb{C}})$ by Lemma 2.4, plays a role in the extension from $Z_0^{\mathbb{C}}$ to $H^{\mathbb{C}}$. Note that this number depends on the Lie group H and thus varies for the different (G,H) that define the same symmetric pair $(\mathfrak{g},\mathfrak{h})$.

Lemma 2.38. *Let $q \in \mathbb{Q}$. The character $q \cdot \chi_{T|\mathfrak{z}^{\mathbb{C}}} \in (\mathfrak{z}^{\mathbb{C}})^*$ lifts to $Z_0^{\mathbb{C}}$ if and only if*

$$\frac{q \cdot \dim \mathfrak{m} \cdot o(e^{2\pi J})}{N} \in \mathbb{Z}.$$

Proof. Suppose that the character $q \cdot \chi_T$ lifts to $Z_0^{\mathbb{C}}$ and call $\hat{\chi}$ this lifting. Since $\alpha(J) = i$ for positive non-compact roots, we have that

$$\hat{\chi}(e^{\theta J}) = e^{\frac{q \cdot \dim \mathfrak{m}}{N} \cdot \theta i}.$$

Consider the diagram

$$\begin{tikzcd}
\mathfrak{z}^{\mathbb{C}} \arrow[r, "q \cdot \chi_{T|\mathfrak{z}^{\mathbb{C}}}"] \arrow[d, "\exp"] & \mathbb{C} \arrow[d, "\exp"] \\
Z_0^{\mathbb{C}} \arrow[r, "\hat{\chi}"] & \mathbb{C}^*.
\end{tikzcd}$$

Denote by Id the identity element of $Z_0^{\mathbb{C}}$. The existence of the lifting is equivalent to $q \cdot \chi_{T|\mathfrak{z}^{\mathbb{C}}}(X) \in 2\pi i\mathbb{Z}$ for all $X \in \exp^{-1}(\mathrm{Id})$. We work with the group G to determine the elements $X \in \mathfrak{z}$ such that $\exp X = \mathrm{Id}$. The element $e^{2\pi J}$ belongs to $Z(G)$, which is a finite group. So, $e^{2\pi J \cdot o(e^{2\pi J})} = \mathrm{Id}$, where $o(e^{2\pi J}) \in \mathbb{Z}$ is the order of the element. The integer multiples of $2\pi J \cdot o(e^{2\pi J})$ are precisely the elements exponentiating to Id. For these, we have:

$$q \cdot \chi_T(2\pi J \cdot o(e^{2\pi J})) = 2\pi \frac{q \cdot \dim \mathfrak{m} \cdot o(e^{2\pi J})}{N} i,$$

and the condition of the lemma follows. □

In Tables 2.1, 2.2 and 2.3, we see that the character χ_T lifts to $Z_0^{\mathbb{C}}$ for the classical and exceptional groups. For the lifting to the group $H^{\mathbb{C}}$ the stronger condition below must be satisfied.

Proposition 2.39. *Define $l := |Z_0^{\mathbb{C}} \cap [H^{\mathbb{C}}, H^{\mathbb{C}}]|$. Let $q \in \mathbb{Q}$. The character $q \cdot \chi_T$ lifts to $H^{\mathbb{C}}$ if and only if*

$$\frac{q \cdot \dim \mathfrak{m} \cdot o(e^{2\pi J})}{l \cdot N} \in \mathbb{Z}.$$

Proof. Let us first recall from [AB83] how every character of $H^{\mathbb{C}}$ is determined by some character $\chi_{T|Z_0^{\mathbb{C}}}$ of $Z_0^{\mathbb{C}}$. Consider the commutator $[H^{\mathbb{C}}, H^{\mathbb{C}}]$, which is the maximal connected semisimple subgroup of $H^{\mathbb{C}}$, and the finite subgroup $D = Z_0^{\mathbb{C}} \cap [H^{\mathbb{C}}, H^{\mathbb{C}}]$. We have $H^{\mathbb{C}} = [H^{\mathbb{C}}, H^{\mathbb{C}}] \times_D Z_0^{\mathbb{C}}$ and the following diagram

$$\begin{tikzcd}
D \arrow[r] & {[H^{\mathbb{C}}, H^{\mathbb{C}}] \times Z_0^{\mathbb{C}}} \arrow[r, "\pi"] \arrow[dr, "\bar{\chi}"] & H^{\mathbb{C}} \arrow[d, "\chi"] \\
 & & \mathbb{C}^{\times}.
\end{tikzcd}$$

Every character of $H^{\mathbb{C}}$ comes from a character of the product $[H^{\mathbb{C}}, H^{\mathbb{C}}] \times Z_0^{\mathbb{C}}$ such that D is contained in its kernel. Since $[H^{\mathbb{C}}, H^{\mathbb{C}}]$ is semisimple, the character $\bar{\chi}$ is defined by a character $Z_0^{\mathbb{C}} \to \mathbb{C}^\times$ which factors through $Z_0^{\mathbb{C}}/D$.

Since G/H is irreducible, $Z^{\mathbb{C}} \cong \mathbb{C}^\times$, and therefore $Z_0^{\mathbb{C}} \cong \mathbb{C}^\times$. As the subgroup $Z_0^{\mathbb{C}} \cap [H^{\mathbb{C}}, H^{\mathbb{C}}]$ is contained in $Z([H^{\mathbb{C}}, H^{\mathbb{C}}])$, it is a finite subgroup of $Z_0^{\mathbb{C}}$, and hence, cyclic. Let $e^{\theta_1 J} \in Z_0^{\mathbb{C}} \cap [H^{\mathbb{C}}, H^{\mathbb{C}}]$ be a generator. Then, for $l = |Z_0^{\mathbb{C}} \cap [H^{\mathbb{C}}, H^{\mathbb{C}}]|$,

$$(e^{\theta_1 J})^l = e^{2\pi J \cdot o(e^{2\pi J})},$$

and therefore, as $e^{\theta_1 J}$ is a generator,

$$\theta_1 = 2\pi \frac{o(e^{2\pi J})}{l}.$$

Analogously to the proof of Lemma 2.38, the condition that D is in the kernel of the character gives the numerical condition. $\square$

Let $q_T \cdot \chi_T$ be the smallest positive rational multiple of χ_T that lifts to $H^{\mathbb{C}}$. We have that

$$q_T = \frac{l \cdot N}{\dim \mathfrak{m} \cdot o(e^{2\pi J})}, \tag{2.39.1}$$

$$q_T \cdot \chi_T = \frac{2l}{\dim \mathfrak{m} \cdot o(e^{2\pi J})} \sum_{\alpha \in \Delta_Q^+} \alpha. \tag{2.39.2}$$

Let $\widetilde{\chi}_T$ be the lifting of $q_T \cdot \chi_T$ to the group $H^{\mathbb{C}}$. In Tables 2.1 and 2.2 we see that $q_T = 1/2$ for all the classical groups except for SO^*, for which $q_T = 1$. In all of these cases, the character lifts to $H^{\mathbb{C}}$. However, for the adjoint groups in Table 2.4, we see that it does not lift to $H^{\mathbb{C}}$.

Remark 2.40. The order of $e^{2\pi J}$ does not necessarily coincide with $|Z(G^{\mathbb{C}}) \cap Z_0^{\mathbb{C}}|$, since this group may not be cyclic. For the adjoint groups, as $o(e^{2\pi J}) = 1$, we need a bigger rational multiple in order to lift the character to the group.

We end this section by proving an additional feature about the Toledo character.

Lemma 2.41. *The Toledo character χ_T defines a Kähler form on G/H by the G-invariant extension of*

$$\omega(Y, Z) = \chi_T([Y, Z]), \text{ for } Y, Z \in \mathfrak{m}.$$

Proof. The form defined is clearly skew-symmetric and invariant. It only remains to show that it is closed, but this is a consequence of G-invariance. In fact, every invariant p-form ω on a symmetric space is closed, as pointed out in [FKK+00], p. 198. Denoting by s the geodesic symmetry at eH, we have that $s\omega$ is also G-invariant and satisfies $s\omega = (-1)^p\omega$. For $d\omega$, we then have $sd\omega = d(s\omega) = (-1)^p d\omega$. But on the other hand we have $sd\omega = (-1)^{p+1}\omega$, so $d\omega = 0$. $\square$

G	H	N	$\dim \mathfrak{m}$	l	$o(e^{2\pi J})$	q_T
$\mathrm{SU}(p,p)$	$\mathrm{S}(\mathrm{U}(p)\times \mathrm{U}(p))$	$2p$	$2p^2$	p	2	$1/2$
$\mathrm{Sp}(2n,\mathbb{R})$	$\mathrm{U}(n)$	$n+1$	$n(n+1)$	n	2	$1/2$
$\mathrm{SO}^*(2n=4m)$	$\mathrm{U}(n)$	$2(n-1)$	$n(n-1)$	n	2	1
$\mathrm{SO}_0(2,n=2m)$	$\mathrm{SO}(2)\times \mathrm{SO}(n)$	n	$2n$	1	1	$1/2$
$\mathrm{SO}_0(2,n=2m+1)$	$\mathrm{SO}(2)\times \mathrm{SO}(n)$	n	$2n$	1	1	$1/2$

Table 2.1: Toledo character data for the classical groups of tube-type

G	H	N	$\dim \mathfrak{m}$	l	$o(e^{2\pi J})$	q_T
$\mathrm{SU}(p,q)$	$\mathrm{S}(\mathrm{U}(p)\times \mathrm{U}(q))$	$p+q$	$2pq$	$lcm(p,q)$	$\frac{p+q}{gcd(p,q)}$	$1/2$
$\mathrm{SO}^*(2n=4m+2)$	$\mathrm{U}(n)$	$2(n-1)$	$n(n-1)$	n	2	1

Table 2.2: Toledo character data for the classical groups of non-tube type.

G	H	N	$\dim \mathfrak{m}$	l	$o(e^{2\pi J})$	q_T
E_7^{-25}	$\mathrm{E}_6^{-78}\times_{\mathbb{Z}_3}\mathrm{U}(1)$	18	54	3	2	$1/2$
E_6^{-14}	$\mathrm{Spin}(10)\times_{\mathbb{Z}_4}\mathrm{U}(1)$	12	32	4	3	$1/2$

Table 2.3: Toledo character data for exceptional groups.

2.4.2 Jordan algebra structure

In this section we endow $\mathfrak{m}_T^{\pm}$ with a Jordan algebra structure. To do this, we endow the real form $\mathfrak{n}_T^{\pm}$ with such a structure and then complexify it. For simplicity, we

G	H	N	$\dim \mathfrak{m}$	l	$o(e^{2\pi J})$	q_T
$\mathrm{PSU}(p,q)$	$\mathrm{PS}(\mathrm{U}(p)\times\mathrm{U}(q))$	$p+q$	$2pq$	$gcd(p,q)$	1	$\frac{p+q}{lcm(p,q)}$
$P\mathrm{SO}^*(2n=4m+2)$	$\mathrm{U}(n)$	$2(n-1)$	$n(n-1)$	n	1	2
E_6^{-14}	$\mathrm{Spin}(10)\times_{\mathbb{Z}_4}\mathrm{U}(1)$	12	32	4	3	$3/2$

Table 2.4: Toledo character data for groups of adjoint type.

work only with $\mathfrak{m}_T^+$ and $\mathfrak{n}_T^+$. As before, the results extend for $\mathfrak{m}_T^-$ and $\mathfrak{n}_T^-$.

We recall that an algebra A over a field F is said to be a **Jordan algebra** if it is commutative ($xy = yx$) and $x(x^2y) = x^2(xy)$ for $x, y \in A$. For $x \in A$, let $L(x)$ denote the linear map $L(x)y = xy$. A Jordan algebra over $\mathbb{R}$ with unit e is said to be Euclidean if it is endowed with a positive definite symmetric bilinear form on A which is associative, i.e., an inner product $(.|.)$ such that $(L(x)u|v) = (u|L(x)v)$. A Jordan algebra is said to be simple if it does not contain any nontrivial ideal.

Given a symmetric cone Ω (in the sense of Definition 2.20) in a Euclidean vector space V, we can endow V with a Jordan algebra structure, as described in [FKK$^+$00], pp. 49-51. This structure depends on the choice of a point in the cone, the stabilizer of which must equal the orthogonal transformations of the cone. This point becomes the identity of the Jordan algebra. The correspondence was originally proved independently by Koecher ([Koe99]) and Vinberg ([Vin60]).

In our case, the Jordan algebra structure on $\mathfrak{n}_T^+$ will stem from the cone $\Omega = H_T^* \cdot e_\Gamma \subset \mathfrak{n}_T^+$ (Lemma 2.23), and the choice of the identity element e_Γ, as follows. We have that the algebra $\mathfrak{h}_T^*$ decomposes into the subalgebra $\mathfrak{h}'$ (annihilator of e_Γ) and the subspace $i\mathfrak{m}'$, which is $\mathrm{Ad}(H_0')$-equivariantly isomorphic to $\mathfrak{n}_T^+$ by the map $X \mapsto \mathrm{ad}(e_\Gamma)(X)$ (Lemma 2.18). This isomorphism is used to define the Jordan product. Given $x_1, x_2 \in \mathfrak{n}_T^-$, let q_1, q_2 be such that $x_j = \mathrm{ad}(q_j)e_\Gamma$ and define $x_1 \cdot x_2 = \mathrm{ad}(q_1)\,\mathrm{ad}(q_2)e_\Gamma$. The element e_Γ is clearly the identity, and the commutativity is easy to check:

$$\mathrm{ad}(q_1)\,\mathrm{ad}(q_2)e_\Gamma = \mathrm{ad}(q_2)\,\mathrm{ad}(q_1)e_\Gamma + \mathrm{ad}(\mathrm{ad}(q_1)q_2)e_\Gamma = \mathrm{ad}(q_2)\,\mathrm{ad}(q_1)e_\Gamma,$$

since $\mathrm{ad}(q_1)q_2 = [q_1, q_2]$ belongs to $[i\mathfrak{m}', i\mathfrak{m}'] = \mathfrak{h}'$, the annihilator of e_Γ. For the second property, $x(x^2y) = x^2(xy)$, we refer to the computations in [FK94], p. 50.

Moreover, we have that some ingredients from Lie theory play a special role in the Jordan algebra structure. The rank of the symmetric space coincides with the rank of the Jordan algebra and the elements $e_{\gamma_1}, \dots, e_{\gamma_r}$ form a Jordan frame, i.e., they generate the algebra and satisfy $e_{\gamma_j}^2 = 1$ and $e_{\gamma_i}e_{\gamma_j} = 0$ for $i \neq j$.

A key ingredient for the present work is the determinant of a Jordan algebra ([FK94], pp. 28-29). We call an element regular if the degree of its minimal polynomial is maximal, and equal to the rank r of the Jordan algebra. The subset of regular elements is an open and dense subset. The minimal polynomial for regular elements x is shown to be given by

$$f(\lambda, x) = \lambda^r - a_1(x)\lambda^{r-1} + a_2(x)\lambda^{r-2} + \ldots + (-1)^r a_r(x)$$

where the $a_j(x)$ are homogeneous polynomials of degree j. The determinant is defined by $\det(x) = a_r(x)$ for all $x \in V$. Note that the determinant is the extension of the polynomial $a_r(x)$ from the regular elements to the whole algebra. By the properties of Jordan frames, we have that if $x = \sum_1^r \lambda_j e_{\gamma_j}$, then $\det(x) = \lambda_1 \cdot \ldots \cdot \lambda_r$.

The following lemma states the semi-invariance of the determinant by the action of the automorphisms of the cone.

Lemma 2.42. *([FK94], III.4.3) Let* det *be the determinant in the Jordan algebra, let $g \in G(\Omega)$ and let* Det *be the determinant of g as an element of* $\mathrm{GL}(V)$*. For $x \in \Omega$, we have that*

$$\det(gx) = \mathrm{Det}(g)^{r/n_T} \det(x).$$

In our situation, the group $G(\Omega)$ is given by H_T^* and the determinant of $h \in H_T^*$ as an element of $\mathrm{GL}(\mathfrak{n}_T^+)$ is given by the determinant of the Adjoint action $\mathrm{Ad}_{\mathfrak{n}_T^+} h$. Note that for the semisimple elements $[H_T^*, H_T^*] \subset H_T^*$ this determinant is trivial, so it defines a character χ^* on H_T^* by

$$\chi^*(h) = \mathrm{Det}(\mathrm{Ad}_{\mathfrak{n}_T^+} h)^{r/n_T}.$$

Reformulating the lemma, we have that for $h \in H_T^*$, and $x \in \Omega$,

$$\det(h \cdot x) = \chi^*(h) \det(x). \tag{2.42.1}$$

Example 2.43. *For $G = \mathrm{SU}(p, q)$, the Lie algebra $\mathfrak{g}^{\mathbb{C}}$ equals $\mathfrak{sl}(p+q, \mathbb{C})$. The subalgebra $\mathfrak{m}^+$ consist of bottom-left blocks of dimension $q \times p$. The decomposition $\mathfrak{m}^+ = \mathfrak{m}_T^+ + \mathfrak{m}_2^+$ is given by the splitting into a $q \times q$ block and a $q \times (q-p)$ block. The real form $\mathfrak{n}_T^+$ is given by these $q \times q$ matrices with real entries. The Jordan algebra determinant for $\mathfrak{n}_T^+$ is the usual Determinant of the matrix B. Let $h = diag(A, D, 1) \in H_T^*$, the adjoint action on $\mathfrak{m}_T^+$ is given by the following product of $(p, p, q-p)$ block-matrices:*

$$\left(\begin{array}{c|cc} A & & \\ \hline & D & \\ & & 1 \end{array}\right) \left(\begin{array}{c|c} & \\ \hline B & \\ 0 & \end{array}\right) \left(\begin{array}{c|cc} A^{-1} & & \\ \hline & D^{-1} & \\ & & 1 \end{array}\right) = \left(\begin{array}{c|c} & \\ \hline DBA^{-1} & \\ 0 & \end{array}\right).$$

Since $\det(\mathrm{Ad}(h)B) = \mathrm{Det}(DBA^{-1}) = \mathrm{Det}(A^{-1})\,\mathrm{Det}(D)\det(B)$, *we have that the character* χ^* *is given by*

$$\chi^*\left(\left(\begin{array}{c|cc} A & & \\ \hline & D & \\ & & 1 \end{array}\right)\right) = \mathrm{Det}(A)^{-1}\,\mathrm{Det}(D).$$

Now, we complexify the Jordan algebra structure on $\mathfrak{n}_T^+$ in the usual way to get a Jordan algebra structure on $\mathfrak{m}_T^+$: $(X+iY)(Z+iT) = XZ - YT + i(YZ+XT)$. The axioms of Jordan algebra are easily verified and the elements $\{e_{\gamma_1}, \dots, e_{\gamma_r}\}$ are again a Jordan frame. The semi-invariance of the determinant 2.42.1 does not extend trivially to the group $H_T^{\mathbb{C}}$. Since the character χ_T^* is defined on the real non-compact group H_T^*, it may not lift to the complexification $H_T^{\mathbb{C}}$. Nonetheless, we have the following result for groups of tube type, in which $H_T = H$, relating the character χ^* and the Toledo character χ_T.

Lemma 2.44. *Let* H *be a group of tube type. The condition for the the Toledo character* χ_T *to lift to* $H^{\mathbb{C}}$ *is equivalent to the condition of the character* χ^* *to be complexified. In that case, the complexification* $\chi^{*,\mathbb{C}}$ *coincides with the lifting* $\widetilde{\chi}_T$.

Proof. If χ^* can be extended to $H^{\mathbb{C}}$, it will be given by $\chi^{*,\mathbb{C}}(h) = \det(\mathrm{Ad}_{\mathfrak{m}^+} h)^{r/n_T}$. Its differential on the Lie algebra $\mathfrak{h}^{\mathbb{C}}$ will be then given by $\frac{r}{n_T}\mathrm{tr}\left(\mathrm{ad}_{\mathfrak{m}^+} X\right)$, for $X \in \mathfrak{h}^{\mathbb{C}}$. We show that this character on $\mathfrak{h}^{\mathbb{C}}$ equals the Toledo character $\chi_T = \frac{2}{N}\sum_{\alpha\in\Lambda_{nc}^+}\alpha$. Since $\mathfrak{m}^+$ is generated by $\{e_\alpha \mid \alpha \in \Lambda_{nc}^+\}$ and $[X, e_\alpha] = \alpha(X)e_\alpha$ for $X \in \mathfrak{t}^{\mathbb{C}}$, we have that for $tJ \in \mathfrak{z}^{\mathbb{C}} \subset \mathfrak{t}^{\mathbb{C}}$,

$$\mathrm{tr}\,\mathrm{ad}_{\mathfrak{m}^+}(tJ) = \sum_{\alpha\in\Lambda_{nc}^+} \alpha(tJ).$$

On the other hand, we have that $\frac{r}{n_T} = \frac{2}{N}$, since $n_T = \frac{1}{2}r(r-1)a + r$ and $N = a(r-1)+2$ in the tube-type case. Thus, the differential of the character χ^* defined on $\mathfrak{h}^{\mathbb{C}}$ always complexifies, since it equals χ_T. Moreover, when χ_T lifts to the character $\widetilde{\chi}_T$ on $H^{\mathbb{C}}$, we have that $\chi^{*,\mathbb{C}}$ exists (χ^* can be complexified) and coincides with $\widetilde{\chi}_T$. □

Remark 2.45. The previous lemma justifies the definition given for the Toledo character χ_T. For groups of tube type, it is the character such that when lifts to $H^{\mathbb{C}}$ describes the semi-invariance of the determinant on $\mathfrak{m}^+$.

We now relate the determinant and the action of the group in general, when the Toledo character does not necessarily lift.

Lemma 2.46. *Let H be of tube type. Let $q \in \mathbb{Q}$ be a positive integer multiple of q_T (see* (2.39.1)*) and recall that $\widetilde{\chi}_T$ is the lifting of $q_T \cdot \chi_T$ to $H^{\mathbb{C}}$. For $h \in H^{\mathbb{C}}$ and $x \in \mathfrak{m}^+$ we have*

$$\det(h \cdot x)^q = \widetilde{\chi}_T(h)^{q/q_T} \det(x)^q. \tag{2.46.1}$$

Proof. Consider the character $(\chi^*)^q$ on H_T^*. This character satisfies

$$\det(h \cdot x)^q = \chi^*(h)^q \det(x)^q \text{ for } h \in \mathrm{H}_T^*.$$

We conclude the same as in Lemma 2.44 since the character lifts to $H^{\mathbb{C}}$ for positive integer multiples of q_T. □

We use the structure of Jordan algebra in $\mathfrak{m}_T^+$ and the action of the group $H^{\mathbb{C}}$ to define a notion of rank on $\mathfrak{m}^+$. Recall that given any system of st-orthogonal roots $\Gamma = \{\gamma_1, \ldots, \gamma_r\}$, by the action of the Cayley transform $\mathrm{Ad}\left(\prod_{\gamma\in\Gamma} \exp(\frac{\pi}{4} i y_\gamma)\right)$ we obtain a subalgebra $\mathfrak{m}_T^+$ which we have just been endowed with a Jordan algebra structure. Polarize the determinant of $\mathfrak{m}_T^+$ to get an r-linear map C such that $C(x, \ldots, x) = \det(x)$. The rank in $\mathfrak{m}_T^+$ is defined as the maximal integer r' such that the $(r-r')$-form $C(\underbrace{x, \ldots, x}_{k}, -, \ldots, -)$ is not identically zero.

Definition 2.47. *Let $\gamma \in \mathfrak{m}^+$. There exists an element $h \in H$ such that $Ad(h)(\gamma)$ lies in $\mathfrak{m}_T^+$. We say that γ has rank r' when the element* $\mathrm{Ad}(h)(\gamma)$ *has rank r'.*

We check that this notion is well defined. On the one hand, we see that such an h exists. By the $\mathrm{Ad}(H)$-equivariant isomorphism $\varphi_+ : \mathfrak{m}^+ \cong \mathfrak{m}$ given by Lemma 2.6 we get an element $\varphi_+(\gamma) \in \mathfrak{m}$. By the fact that $\mathrm{Ad}(H)\mathfrak{a} = \mathfrak{m}$, there exists some $h \in H$ such that $\mathrm{Ad}(h)\varphi_+(\gamma)$ lies in $\mathfrak{a} \subset \mathfrak{m}_T$. Going back to $\mathfrak{m}^+$ with φ_+^{-1}, which is $\mathrm{Ad}(H)$-equivariant, we get an element $\mathrm{Ad}(h)\gamma \in \mathfrak{m}_T^+$ for which the rank is well defined. On the other hand, we see that it is uniquely defined. Given $h_1, h_2 \in H$ such that $\mathrm{Ad}(h_1)\gamma, \mathrm{Ad}(h_2)\gamma \in \mathfrak{m}_T^+$, we have that the ranks of these two elements are the same since they are related by $\mathrm{Ad}(h_1 h_2^{-1})$. The definition does not depend on the choice of Γ as any two systems of orthogonal roots are related by the action of some element of H.

Remark 2.48. In the case of $\mathrm{SU}(p, q)$, this specializes to the notion of rank for a rectangular matrix $q \times p$.

The following proposition plays an important role in what follows.

Proposition 2.49. *Let $1 \leq r' \leq r$. The group $H^{\mathbb{C}}$ acts transitively on the elements of rank r' in $\mathfrak{m}^+$.*

Proof. We use again the $\mathrm{Ad}(H)$-equivariant isomorphism $\varphi_+ : \mathfrak{m}^+ \cong \mathfrak{m}$ given by Lemma 2.6 and the fact that $\mathrm{Ad}(H)\mathfrak{a} = \mathfrak{m}$. For $X \in \mathfrak{m}^+$, take $\varphi(X) \in \mathfrak{m}$ and consider $h \in H$ such that $\mathrm{Ad}(h)\varphi(X) \in \mathfrak{a} = \langle x_\gamma \rangle_{\gamma\in\Gamma}$. Back to $\mathfrak{m}^+$, we have that $\mathrm{Ad}(h)X \in \mathfrak{a}^+$, where $\mathfrak{a}^+ = \langle e_\gamma \rangle_{\gamma\in\Gamma}$ was defined in Lemma 2.12. Thus, it suffices to show the transitivity on $\mathfrak{a}^+$. Recall that

$$\det\left(\sum_{\gamma\in\Gamma} c_\gamma e_\gamma\right) = \prod_{\gamma\in\Gamma} c_\gamma,$$

and hence, elements of rank r' in $\mathfrak{a}^+$ correspond to those $\sum_{\gamma\in\Gamma} c_\gamma e_\gamma$ with exactly r' non-zero coefficients c_γ.

We now distinguish three cases: maximal rank, rank 1 and finally, any other rank. We begin by the elements of maximal rank. Take $e_\Gamma \in \mathfrak{a}^+$. Consider the action of $h = \prod_{\gamma\in\Gamma} \exp(t_\gamma h_\gamma) \in H^{\mathbb{C}}$ with $t_\gamma \in \mathbb{C}$. We have that $\mathrm{Ad}(h)e_\Gamma = \sum_{\gamma\in\Gamma} 2\cdot e^{t_\gamma} e_\gamma \in \mathfrak{a}^+$, and therefore the action is transitive on the elements of maximal rank.

Now take two elements of rank 1 in $\mathfrak{a}^+$, say e_{γ_j} and e_{γ_k} for $k \neq j$. We define an element h such that $\mathrm{Ad}(h)e_{\gamma_j} = e_{\gamma_k}$. Consider a root $\varphi \in C_{jk}$, i.e., a root projecting to $\frac{1}{2}(\gamma_k - \gamma_j)$. Then, $\varphi + \gamma_j$ is a root in Q_{jk} and $\varphi - \gamma_j$ is not a root. We use the notation $x_\alpha = e_\alpha + e_{-\alpha}$ even though α is compact. Since

$$[x_\varphi, e_{\gamma_j}] = [e_\varphi, e_{\gamma_j}] = e_{\varphi+\gamma_j} \qquad [x_\varphi, e_{\varphi+\gamma_j}] = [e_{-\varphi}, e_{\varphi+\gamma_j}] = e_{\gamma_j},$$

we have that $\mathrm{Ad}\left(\exp\left(\frac{\pi}{2}x_\varphi\right)\right)e_{\gamma_j}$ equals

$$\exp\left(\mathrm{ad}\left(\frac{i\pi}{2}x_\varphi\right)\right)e_{\gamma_j} = \cos\left(\frac{\pi}{2}\right)e_{\gamma_j} + \sin\left(\frac{\pi}{2}\right)e_{\varphi+\gamma_j} = e_{\varphi+\gamma_j},$$

where $\varphi + \gamma_j$ is a root in Q_{jk}. Moreover, φ runs over C_{jk} as $\varphi + \gamma_j$ runs over Q_{jk}. Analogously, for $\lambda \in C_{kj}$ we have that

$$\mathrm{Ad}\left(\exp\left(\frac{i\pi}{2}x_\lambda\right)\right)e_{\gamma_k} = e_{\lambda+\gamma_k} \in Q_{jk},$$

and while λ runs over C_{jk}, $\lambda + \gamma_j$ runs over Q_{jk}. Thus, for every $\varphi_0 \in C_{jk}$ there exists a $\lambda_0 \in C_{kj}$ such that

$$\mathrm{Ad}\left(\exp\left(\frac{\pi}{2}i - x_{\lambda_0}\right)\exp\left(\frac{\pi}{2}ix_{\varphi_0}\right)\right)e_{\gamma_j} = e_{\gamma_k}.$$

By the action of $\exp(t_{\gamma_k}h_{\gamma_k})$ we get an arbitrary element $2e^{t_{\gamma_k}}e_{\gamma_k}$.

Finally, for any other rank $1 < r' < r$, combine both techniques. The elements of the form $\exp\left(\frac{\pi}{2}(-x_{\lambda_0} + x_{\varphi_0})\right)$ allow to permute the coefficients c_γ, and $\prod_{\gamma\in\Gamma}\exp(t_\gamma h_\gamma)$ provide the desired coefficients. $\square$

Remark 2.50. In the case of $\mathrm{SU}(2,3)$, the operation performed for the rank 1 elements correspond to

$$\mathrm{Ad}\left(\left(\begin{array}{cc|ccc} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{array}\right)\right)\left(\begin{array}{cc|ccc} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right) = \left(\begin{array}{cc|ccc} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right),$$

if we take $\gamma_j = \gamma_1 = x_3 - x_2$, $\gamma_k = \gamma_2 = x_4 - x_1$, $\varphi = x_1 - x_2$, $\lambda = x_3 - x_4$.

Corollary 2.51. *Let $\mathfrak{m}^+_{D\neq 0}$ be the set of elements of maximal rank in $\mathfrak{m}^+$ and $H'^{\mathbb{C}}$ the stabilizer of e_Γ. Then,*

$$\mathfrak{m}^+_{D\neq 0} \cong \frac{H^{\mathbb{C}}}{H'^{\mathbb{C}}}.$$

We end this section by proving a lemma on the existence of a canonical three-dimensional simple subalgebra containing a given element of $\mathfrak{m}^+$. Recall that a three-dimensional simple subalgebra is a subalgebra isomorphic to $\mathfrak{sl}(2,\mathbb{C})$.

Lemma 2.52. *Let $X \in \mathfrak{m}^+$ and $\varphi^-_+ : \mathfrak{m}^+ \to \mathfrak{m}^-$ be the $\mathrm{Ad}(H^{\mathbb{C}})$-equivariant isomorphism given by Lemma 2.6. The subspace generated by $\langle X, \varphi^-_+(X), [X, \varphi^-_+(X)]\rangle$ is a three-dimensional simple subalgebra.*

Proof. Let r' be the rank of γ. Take $h \in H^{\mathbb{C}}$ such that $\mathrm{Ad}(h)X = e_{\Gamma'}$ for some subset of r' elements Γ' of a system of st-orthogonal roots Γ. It is clear that $\varphi^-_+(\mathrm{Ad}(h)X) = e_{-\Gamma'}$, $[e_{-\Gamma'}, e_\Gamma] = h_\Gamma$ and that $\{e_{-\Gamma'}, e_{\Gamma'}, h_\Gamma\}$ span a three-dimensional simple subalgebra. By $\mathrm{Ad}(H^{\mathbb{C}})$-equivariance we conclude that $\langle X, \varphi^-_+(X), [X, \varphi^-_+(X)]\rangle$ is a three-dimensional simple subalgebra. ☐

Remark 2.53. See Remark B.3 for another approach to the determinant, and see Remarks B.7 and B.8 for more on the definition of rank and the relation with Jordan triples.

2.5 Parabolic subgroups

2.5.1 Basics on R-parabolic subgroups

The notion of parabolic subgroup is usually defined for connected affine algebraic groups G, as in the classic reference [Bor91]. These are the subgroups P such that G/P is a compact variety, or equivalently, those containing a maximal connected solvable subgroup $B \subset G$, called a Borel subgroup. The unipotent radical of P is the maximal unipotent connected solvable normal subgroup of P. A complement for the unipotent radical is called a Levi subgroup, and it is a maximal reductive subgroup of P. Given a Levi subgroup L and $p \in P$, pLp^{-1} is also a Levi subgroup. Since L is not normal when $P \neq G$, a Levi subgroup is not uniquely defined.

When dealing with non-connected groups one can consider Richardson parabolic subgroups or, for the sake of brevity, R-parabolic subgroups. They were named after they appeared in [Ric88] and have been studied in Section 6 of [BMR05] for non-connected groups. Given a one-parameter multiplicative subgroup $\lambda : \mathbb{R}^* \to G$, define $P_\lambda = \{g \in G \mid \lim_{t\to 0} \lambda(t) g \lambda(t)^{-1} \text{ exists }\}$. This is called the R-parabolic subgroup associated with λ.

We focus on the case of a complex reductive group $H^{\mathbb{C}}$, not necessarily connected, with compact real form H. Given an element $s \in i\mathfrak{h}$, consider the multiplicative subgroup $\lambda(\frac{1}{t}) = e^{ts} \in H^{\mathbb{C}}$. Then, the following are R-parabolic and Levi subgroups and subalgebras:

$$
\begin{aligned}
P_s &= \{g \in H^{\mathbb{C}} \ : \ e^{ts} g e^{-ts} \text{ is bounded as } t \to \infty\} \\
L_s &= \{g \in H^{\mathbb{C}} \ : \ \mathrm{Ad}(g)(s) = s\} \\
\mathfrak{p}_s &= \{Y \in \mathfrak{h}^{\mathbb{C}} \ : \ \mathrm{Ad}(e^{ts})Y \text{ is bounded as } t \to \infty\} \\
\mathfrak{l}_s &= \{Y \in \mathfrak{h}^{\mathbb{C}} \ : \ \mathrm{ad}(Y)(s) = [Y, s] = 0\}.
\end{aligned}
$$

When G is connected, the notion of R-parabolic recovers the notion of parabolic subgroup.

For G a real reductive Lie group and $H \subset G$ a maximal compact subgroup, we have the Cartan decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ and the isotropy representation $\mathrm{Ad} : H^{\mathbb{C}} \to \mathrm{Aut}(\mathfrak{m}^{\mathbb{C}})$. Apart from the subgroups and subalgebras of $H^{\mathbb{C}}$ and $\mathfrak{h}^{\mathbb{C}}$ defined above, consider the subspaces

$$
\begin{aligned}
\mathfrak{m}_s &= \{Y \in \mathfrak{m}^{\mathbb{C}} \ : \ \mathrm{Ad}(e^{ts})Y \text{ is bounded as } \ t \to \infty\} \\
\mathfrak{m}_s^0 &= \{Y \in \mathfrak{m}^{\mathbb{C}} \ : \ \mathrm{Ad}(e^{ts})Y = Y \ \text{ for every } \ t\}.
\end{aligned}
$$

One has that $\mathfrak{m}_s$ is invariant under the action of P_s and $\mathfrak{m}_s^0$ is invariant under the action of L_s. The following lemma is helpful to describe the spaces just defined in terms of root vectors.

Remark 2.54. The subalgebra $\mathfrak{m}_s$ is the non-compact part of the parabolic subalgebra of $\mathfrak{g}^{\mathbb{C}}$ defined by $s \in i\mathfrak{h}$. Define

$$
\widetilde{\mathfrak{p}}_s = \{Y \in \mathfrak{g}^{\mathbb{C}} \mid \mathrm{Ad}(e^{ts})Y \text{ is bounded as } t \to \infty\}.
$$

We have that $\mathfrak{p}_s = \widetilde{\mathfrak{p}}_s \cap \mathfrak{h}^{\mathbb{C}}$ and $\mathfrak{m}_s = \widetilde{\mathfrak{p}}_s \cap \mathfrak{m}^{\mathbb{C}}$. Analogously, define

$$\widetilde{\mathfrak{l}}_s = \{Y \in \mathfrak{g}^{\mathbb{C}} \mid \operatorname{Ad}(e^{ts})Y = Y \text{ for every } t\}.$$

We have that $\mathfrak{l}_s = \widetilde{\mathfrak{l}}_s \cap \mathfrak{h}^{\mathbb{C}}$ and $\mathfrak{m}_s^0 = \widetilde{\mathfrak{l}}_s \cap \mathfrak{m}^{\mathbb{C}}$.

Lemma 2.55. *Given $s \in i\mathfrak{h}$, we have that*

$$\begin{aligned}
\mathfrak{p}_s &= \langle Y \in \mathfrak{h}^{\mathbb{C}} \ : \ \operatorname{ad}(s)Y = \lambda_Y Y \ \textit{ for } \lambda_Y \leq 0\rangle \\
\mathfrak{l}_s &= \langle Y \in \mathfrak{h}^{\mathbb{C}} \ : \ \operatorname{ad}(s)Y = 0\rangle \\
\mathfrak{m}_s &= \langle Y \in \mathfrak{m}^{\mathbb{C}} \ : \ \operatorname{ad}(s)Y = \lambda_Y Y \ \textit{ for } \lambda_Y \leq 0\rangle \\
\mathfrak{m}_s^0 &= \langle Y \in \mathfrak{m}^{\mathbb{C}} \ : \ \operatorname{ad}(s)Y = 0\rangle.
\end{aligned}$$

Proof. We consider the endomorphism $\operatorname{ad}(s)$ and take $\{Y_\delta\}_{\delta \in D \subset \mathbb{C}} \subset \mathfrak{m}^{\mathbb{C}}$, a basis of eigenvectors such that $\operatorname{ad}(s)Y_\delta = \delta Y_\delta$. We have that

$$\operatorname{Ad}(e^{ts})Y_\delta = e^{\operatorname{ad}(ts_\chi)}Y_\delta = \sum_{j=0}^{\infty} \frac{(\operatorname{ad}(ts))^j(Y_\delta)}{j!} = \left(\sum_{j=0}^{\infty} \frac{(t\lambda)^j}{j!}\right) Y_\delta = e^{t\lambda}Y_\delta.$$

Therefore, v_δ belongs to $\mathfrak{m}_s$ (resp. $\mathfrak{m}_\chi^0$) if and only if $\lambda \leq 0$ (resp. $\lambda = 0$). By linearity, we obtain the result. □

Recall that a character of a complex Lie algebra $\mathfrak{g}$ is a complex linear map $\mathfrak{g} \to \mathbb{C}$ which factors through the quotient map $\mathfrak{g} \to \mathfrak{g}/[\mathfrak{g}, \mathfrak{g}]$. For a parabolic subalgebra $\mathfrak{p}$, let $\mathfrak{l}$ be a corresponding Levi subalgebra with centre $\mathfrak{z}_L$. One shows that $(\mathfrak{p}/[\mathfrak{p}, \mathfrak{p}])^* \cong \mathfrak{z}_L^*$, and then a character χ of $\mathfrak{p}$ comes from an element in $\mathfrak{z}_L^*$. Using the Killing form of $\mathfrak{g}^{\mathbb{C}}$, which is non-degenerate, from $\chi \in \mathfrak{z}_L^*$ we get an element of $s_\chi \in \mathfrak{z}_L \subset i\mathfrak{h}$. Conversely, any $s \in i\mathfrak{h}$ defines a character χ_s of $\mathfrak{p}_s$ since $B(s, [\mathfrak{p}_s, \mathfrak{p}_s]) = 0$.

When $\mathfrak{p} \subset \mathfrak{p}_{s_\chi}$, we say that χ is an antidominant character of $\mathfrak{p}$. When the equality is attained, $\mathfrak{p} = \mathfrak{p}_{s_\chi}$, we say that χ is a strictly antidominant character. Note that for $s \in i\mathfrak{h}$, χ_s is a strictly antidominant character of $\mathfrak{p}_s$.

Remark 2.56. An approach based on root theory can be found in [GGM09]. There, the antidominant characters are also described in terms of fundamental weights.

2.5.2 Relevant parabolic subgroups

In this section we show that for $m \in \mathfrak{m}^+$ the subspace

$$\mathfrak{p}_m = \operatorname{Ker}(\operatorname{ad}(m)_{|\mathfrak{h}^{\mathbb{C}}}) \oplus \operatorname{Im}(\operatorname{ad}(m)_{|\mathfrak{m}^-})$$

is a parabolic subalgebra of $\mathfrak{h}^{\mathbb{C}}$. To do this, we give an antidominant character of it which will also play an important role later.

We first show that $\mathfrak{p}_m$ is a Lie subalgebra. Consider the three-dimensional simple subalgebra containing m given by Lemma 2.52 and its adjoint action on $\mathfrak{g}^{\mathbb{C}}$. Since $\mathrm{ad}(m)$ has order 3, the only possible irreducible subrepresentations of the adjoint representation are of dimension 1, 2 and 3. By taking a multiple of m, which gives the same subspace $\mathfrak{p}_m$, we may assume that the weights of these representations for the action of m can be normalized to $\{0\}$, $\{-1, 1\}$ and $\{-2, 0, 2\}$ respectively. The distribution of these weight spaces in the decomposition $\mathfrak{g}^{\mathbb{C}} = \mathfrak{h}^{\mathbb{C}} + \mathfrak{m}^+ + \mathfrak{m}^-$ is as follows:

$$\begin{tikzcd}
\mathfrak{m}^- = & -2 \arrow[d, "\mathrm{ad}(m)"] & -1 \arrow[d, "\mathrm{ad}(m)"] & & 0 \\
\mathfrak{h}^{\mathbb{C}} = & 0 \arrow[d, "\mathrm{ad}(m)"] & 1 & -1 \arrow[d, "\mathrm{ad}(m)"] & 0 \\
\mathfrak{m}^+ = & 2 & & 1 & 0.
\end{tikzcd} \tag{2.56.1}$$

The circled subspaces correspond to those weight spaces contained in $\mathfrak{p}_m$. As the Lie bracket of elements of weight 0 and -1 never reach elements of weight 1, we have that $\mathfrak{p}_m$ is indeed a subalgebra.

We now fix a Cartan subalgebra $\mathfrak{t}^{\mathbb{C}} \subset \mathfrak{h}^{\mathbb{C}}$ and a system of st-orthogonal roots $\Gamma = \{\gamma_1, \ldots, \gamma_r\} \subset \Delta(\mathfrak{g}^{\mathbb{C}}, \mathfrak{t}^{\mathbb{C}})$. Take $1 \leq r' \leq r$ and let $\Gamma' = \{\gamma_1, \ldots, \gamma_{r'}\}$. Consider the element of rank r'

$$e_{\Gamma'} = e_{\gamma_1} + \ldots + e_{\gamma_{r'}} \in \mathfrak{m}^+.$$

We study the subalgebra $\mathfrak{p}_{e_{\Gamma'}}$. To prove that it is parabolic we define a strictly antidominant character χ' with dual $s' := s_{\chi'} \in i\mathfrak{h}$ such that $\mathfrak{p}_{e_{\Gamma'}} = \mathfrak{p}_{s'}$. Below, we will extend our results to the whole $\mathfrak{m}^+$ by $\mathrm{Ad}(H^{\mathbb{C}})$-equivariance.

The theory of restricted roots will help us to calculate $\mathfrak{p}_{s'}$. Every root in $\Delta(\mathfrak{g}^{\mathbb{C}}, \mathfrak{t}^{\mathbb{C}})$ restricts either to 0, $\pm\gamma_i$, $\pm\frac{1}{2}(\gamma_i \pm \gamma_j)$, or $\pm\frac{1}{2}\gamma_i$. In these cases we say that the root belongs to C_0, $\pm\Gamma$, $\pm C_{ij}(\pm Q_{ij})$, or $\mp C_i(\pm Q_i)$ respectively (recall Section 2.3.1). For simplicity, we put together some of these subsets of roots, depending on whether the indices are less than or equal to, or greater than r'. For instance, we put together all the compact roots restricting to $\frac{1}{2}(\gamma_i - \gamma_j)$ for $1 \leq i \leq r'$ and $r' < j \leq r$ and we call this set $C_{><}$, referring to the fact that the first index is less or equal than r' and the second is greater than r'. We get the following subsets (and their opposites in some cases).

$$C_{<<} = \bigcup_{1\le i,j\le r'} C_{ij} \quad C_{>>} = \bigcup_{r'<i,j\le r} C_{ij} \quad Q_{<<} = \bigcup_{1\le i,j\le r'} Q_{ij} \quad Q_{>>} = \bigcup_{r'<i,j\le r} Q_{ij}$$

$$C_{><} = \bigcup_{\substack{1\le i\le r' \\ r'<j\le r}} C_{ij} \quad C_{<>} = \bigcup_{\substack{1\le j\le r' \\ r'<i\le r}} C_{ij} \quad Q_{<>} = \bigcup_{\substack{1\le i\le r' \\ r'<j\le r}} Q_{ij} \quad Q_{><} = \bigcup_{\substack{1\le j\le r' \\ r'<i\le r}} Q_{ij}$$

$$C_{<} = \bigcup_{1\le i\le r'} C_{i} \qquad C_{>} = \bigcup_{r'<i\le r} C_{i} \qquad Q_{<} = \bigcup_{1\le i\le r'} Q_{i} \qquad Q_{>} = \bigcup_{r'<i\le r} Q_{i}$$

The case of $\mathfrak{su}(p,q)$ will help us to put all this information in a graphical way. Take the Cartan subalgebra of diagonal matrices $\mathfrak{t}^{\mathbb{C}}$ inside $\mathfrak{sl}(p+q,\mathbb{C})$, let $\{x_i-x_j\}_{1\le i\ne j\le p+q}$ be the root system, and let $\Gamma = \{\gamma_j = x_{p+j}-x_{p-j+1}\}_{1\le j\le r}$ be a st-orthogonal system. Off-diagonal matrix units (matrices with only one off-diagonal non-zero entry) E_{ij} are generators of the root spaces. Forgetting about the diagonal, which represents $\mathfrak{t}^{\mathbb{C}}$, we have that every entry of the matrix represents a root. If we label that entry with the subset which it belongs to, we obtain the following matrix, with blocks C_* and Q_*:

$$M = \left(\begin{array}{cc|ccc} C_{>>} & C_{><} & -Q_{<>} & -Q_{>>} & -Q_{>} \\ C_{<>} & C_{<<} & -Q_{<<} & -Q_{<>} & -Q_{<} \\ \hline Q_{<>} & Q_{<<} & C_{<<} & C_{><} & C_{<} \\ Q_{>>} & Q_{<>} & C_{<>} & C_{>>} & C_{>} \\ Q_{>} & Q_{<} & -C_{<} & -C_{>} & C_{0} \end{array}\right),$$

where the sizes of the blocks are determined by partitioning the rows of M into sets of size $(p-r', r', r', q-r', q-p)$, and then partitioning the columns in the same way. Note that there are some subsets of roots repeated, since the multiplicity of the roots projecting to $\pm\gamma_i \pm \gamma_j$ is 2 in the case of $\mathfrak{su}(p,q)$.

We adopt one more convention. For subsets of roots $T_1,\dots,T_k \subset \Delta(\mathfrak{g}^{\mathbb{C}},\mathfrak{t}^{\mathbb{C}})$, we define the vector space which is generated by the root spaces corresponding to their roots:

$$\mathfrak{g}^{\mathbb{C}}[T_1,\dots,T_j] = \bigcup_{\alpha\in T_1} \mathfrak{g}^{\mathbb{C}}_{\alpha} \cup \ldots \cup \bigcup_{\alpha\in T_k} \mathfrak{g}^{\mathbb{C}}_{\alpha}.$$

The following proposition describes $\mathfrak{p}_{e_{\Gamma'}}$ in terms of these subsets of roots.

Lemma 2.57. *We have that*

$$\begin{aligned}\operatorname{Ker}(\operatorname{ad}(e_{\Gamma'})_{|\mathfrak{h}^{\mathbb{C}}}) =& \mathfrak{g}^{\mathbb{C}}[C_{<>}, C_{>>}, \pm C_{>}, C_{<}] \cup \langle e_\alpha \mid \alpha \in C_0\rangle \\ & \cup \langle e_\alpha - e_\beta \mid -\alpha, \beta \in C_{ij} \textit{ for } 1 \le i, j \le r'\rangle \\ \operatorname{Im}(\operatorname{ad}(e_{\Gamma'})_{|\mathfrak{m}^+}) =& \langle e_\alpha + e_\beta \mid -\alpha, \beta \in C_{ij} \textit{ for } 1 \le i, j \le r'\rangle \cup \mathfrak{t}^{\mathbb{C}}.\end{aligned}$$

Hence, $\mathfrak{p}_{e_{\Gamma'}} = \operatorname{Ker}(\operatorname{ad}(e_{\Gamma'})_{|\mathfrak{h}^{\mathbb{C}}}) \oplus \operatorname{Im}(\operatorname{ad}(e_{\Gamma'})_{|\mathfrak{m}^+}) = \mathfrak{g}^{\mathbb{C}}[C_{<<}, C_{<>}, C_{>>}, \pm C_{>}, C_{<}] \cup \mathfrak{t}^{\mathbb{C}}$.

Proof. It follows directly from the expression of $e_{\Gamma'}$, the decomposition of $\mathfrak{g}^{\mathbb{C}}$ into root spaces and the fact that $[e_\alpha, e_\beta] = e_{\alpha+\beta}$ when $\alpha + \beta$ is a root or 0 otherwise. For the last statement, we have that the sum of the subspaces $\langle e_\alpha - e_\beta\rangle$ and $\langle e_\alpha - e_\beta\rangle$ with $-\alpha, \beta \in C_{ij}$ for $1 \le i, j \le r'$ gives $\langle e_\alpha \mid \alpha \in C_{ij}$ for $1 \le i, j \le r'\rangle$. □

By similar arguments to those used in the previous proof we get the following.

Lemma 2.58. *The subspace* $\operatorname{Im}((\operatorname{ad} e_{\Gamma'})^2)$ *equals* $\mathfrak{g}^{\mathbb{C}}[Q_{<<}]$*, and hence,* $e_{\Gamma'} \in \operatorname{Im}((\operatorname{ad} e_{\Gamma'})^2)$.

We now modify the Toledo character to get an antidominant character of $\mathfrak{p}_{r'}$. We define $\chi_{\Gamma'} = \sum_{\gamma \in \Gamma'} \gamma$ and

$$\chi' = \chi_T - \chi_{\Gamma'}.$$

Remark 2.59. By Remark 2.36, the new summand $\chi_{\Gamma'}$ would correspond to the Toledo character of a tube with root system

$$\Delta_{\Gamma'} := \bigcup \{Q_{ij} \cup C_{ij} \mid 1 \le i, j \le r'\} \cup \{\gamma_1, \dots, \gamma_{r'}\} \subset \Delta(\mathfrak{g}^{\mathbb{C}}, \mathfrak{t}^{\mathbb{C}}),$$

with the same notions of positivity and compactness as $\Delta(\mathfrak{g}^{\mathbb{C}}, \mathfrak{t}^{\mathbb{C}})$.

As usual, let $s' = s_{\chi'}$ be the dual of the character χ' with respect to the Killing form B. By Lemma 2.55, to determine the parabolic subalgebra $\mathfrak{p}_{s_\chi}$ for any character χ it is enough to study the action $\operatorname{ad}(s_\chi)$ on root vectors e_α. The subalgebra $\mathfrak{p}_{s_\chi}$ is then generated by those elements of non-positive eigenvalue. The action is given by

$$\operatorname{ad}(s_\chi) e_\alpha = \alpha(s_\chi) e_\alpha = \chi(h_\alpha) e_\alpha$$

and it is thus completely determined by $\chi(h_\alpha) = \alpha(s_\chi)$.

In the case of χ_T and $\chi_{\Gamma'}$, and therefore χ', the value at the elements h_α only depends on the subset of roots C_*, Q_* to which α belongs, so we can summarize the information of a character using the matrix M defined above, and putting in the entry corresponding to C_* or Q_*, the value of the character at the elements h_α such that $\alpha \in C_*, Q_*$.

For instance, from Lemmas 2.33 and 2.34, we have that χ_T acts as $\sum_j \gamma_j$ on h_α and hence,

$$\chi_T(h_\alpha) = \begin{cases} 0 & \text{for } \alpha \in C_{<<}, C_{<>}, C_{><}, C_{>>}, \pm C_<, \pm C_>, C_0 \\ 2 & \text{for } \alpha \in Q_{<<}, Q_{<>}, Q_{>>}, Q_<, Q_> \\ -2 & \text{for } \alpha \in -Q_{<<}, -Q_{<>}, -Q_{>>}, -Q_<, -Q_> \end{cases}.$$

What we write as

$$M\left(-\chi_T\right) = \left(\begin{array}{cc|ccc} 0 & 0 & -2 & -2 & -2 \\ 0 & 0 & -2 & -2 & -2 \\ \hline 2 & 2 & 0 & 0 & 0 \\ 2 & 2 & 0 & 0 & 0 \\ 2 & 2 & 0 & 0 & 0 \end{array}\right).$$

In the case of $\chi_{\Gamma'} = \sum_{\gamma \in \Gamma'} \gamma$, we have

$$\chi_{\Gamma'}(h_\alpha) = \begin{cases} 0 & \text{for } \alpha \in C_{<<}, C_{>>}, \pm C_>, C_0, \pm Q_{>>}, \pm Q_> \\ -1 & \text{for } \alpha \in C_{<>}, C_<, -Q_{<>}, -Q_< \\ 1 & \text{for } \alpha \in C_{><}, -C_<, Q_{<>}, Q_< \\ 2 & \text{for } \alpha \in Q_{<<} \\ -2 & \text{for } \alpha \in Q_{>>} \end{cases}$$

and the corresponding matrix

$$M\left(\chi_{\Gamma'}\right) = \left(\begin{array}{cc|ccc} 0 & -1 & 1 & 0 & 0 \\ 1 & 0 & 2 & 1 & 1 \\ \hline -1 & -2 & 0 & -1 & -1 \\ 0 & -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 \end{array}\right).$$

Proposition 2.60. *The subalgebra $\mathfrak{p}_{e_{\Gamma'}}$ coincides with $\mathfrak{p}_{s'}$, and it is therefore parabolic. We also have that $\mathfrak{m}^0_{s'} \cap \mathfrak{m}^+ = \mathrm{Im}(\mathrm{ad}(e_{\Gamma'})^2)$ and $\mathfrak{m}_{s'} = \mathfrak{m}^- \oplus \mathfrak{m}^0_{s'} \cap \mathfrak{m}^+$.*

Proof. We use the previous examples to get the values of the character $\chi' = \chi_T - \chi_{\Gamma'}$ at h_α:

$$\chi'(h_\alpha) = \begin{cases} 0 & \text{for } \alpha \in C_{<<}, \pm Q_{<<}, C_{>>}, \pm C_>, C_0 \\ -1 & \text{for } \alpha \in C_{<>}, C_<, Q_{<>}, Q_< \\ 1 & \text{for } \alpha \in C_{><}, -C_<, -Q_{<>}, -Q_< \\ 2 & \text{for } \alpha \in Q_{>>}, Q_> \\ -2 & \text{for } \alpha \in -Q_{>>}, -Q_> \end{cases}$$

This can be represented as the matrix

$$M(\chi') = \left(\begin{array}{cc|ccc} 0 & -1 & -1 & -2 & -2 \\ 1 & 0 & 0 & -1 & -1 \\ \hline 1 & 0 & 0 & -1 & -1 \\ 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \end{array}\right).$$

Thus, the roots with non-positive eigenvalue are the same as in Lemma 2.57, and $\mathfrak{p}_{e_{\Gamma'}} = \mathfrak{p}_{s'}$. Futhermore, we see that $\mathfrak{m}^0_{\chi'}$ equals $\mathfrak{g}^{\mathbb{C}}[\pm Q_{<<}]$. So, $\mathfrak{m}^0_{\chi'} \cap \mathfrak{m}^+ = \mathfrak{g}^{\mathbb{C}}[Q_{<<}]$, which equals $\mathrm{Im}((\operatorname{ad} e_\Gamma)^2)$ by Lemma 2.58, and $\mathfrak{m}_{\chi'} = \mathfrak{m}^- + \mathfrak{m}^0_{\chi'} \cap \mathfrak{m}^+$ with $\mathfrak{m}^- \cap (\mathfrak{m}^0_{\chi'} \cap \mathfrak{m}^+) = \emptyset$. □

Remark 2.61. Note that since the case of $\mathfrak{su}(p,q)$ has been the model for the matrix M, we do obtain the shape of the parabolic $\mathfrak{p}_{s'}$ in the case of $\mathfrak{su}(p,q)$ by looking at the non-positive entries of the matrix $M(\chi)$.

We now generalize this result to any element $m \in \mathfrak{m}^+$.

Proposition 2.62. *The subalgebra $\mathfrak{p}_m$ for $m \in \mathfrak{m}^+$ is a parabolic subalgebra.*

Proof. Given a system of st-orthognal roots Γ and a subset of r' elements $\Gamma' \subset \Gamma$, we have defined a parabolic subalgebra $\mathfrak{p}_{e_{\Gamma'}}$ and an antidominant character χ'. Every element $m \in \mathfrak{m}^+$ is conjugate to one of these $e_{\Gamma'}$, since the group $H^{\mathbb{C}}$ acts transitively in the elements of given rank (Lemma 2.49). Thus, the subalgebra $\mathfrak{p}_m$ is isomorphic to some $\mathrm{Ad}(h)\mathfrak{p}_{e_{\Gamma'}}$ for some $h \in H^{\mathbb{C}}$ and it is indeed a parabolic subalgebra. We also obtain an antidominant character given by $\chi \circ \mathrm{Ad}(h)$. □

We also have the following lemma, which can be proved from Lemma 2.58 by $H^{\mathbb{C}}$-equivariance, but we refer to Lemma 3.3 of [Kos59] for a general proof based on the nilpotency of $\mathrm{ad}(m)$.

Lemma 2.63. *Let $m \in \mathfrak{m}^+$. Then, $m \in \mathrm{Im}((\operatorname{ad} m)^2)$.*

Let $P_{e_{\Gamma'}}$ be the connected subgroup of $H^{\mathbb{C}}$ corresponding to the Lie subalgebra $\mathfrak{p}_{e_{\Gamma'}} \subset \mathfrak{h}^{\mathbb{C}}$. As a consequence of Proposition 2.60, we have that $P_{e_{\Gamma'}}$ equals $P_{s'}$, and it is therefore a parabolic subgroup. We extend this result to $\mathfrak{m}^+$ by the action of $\mathrm{Ad}(H^{\mathbb{C}})$ and get that P_m, the subgroup of $H^{\mathbb{C}}$ corresponding to $\mathfrak{p}_m \subset \mathfrak{h}^{\mathbb{C}}$, is a parabolic subgroup. A character χ_m may not lift to P_m, but some rational multiple must do it. Let q_m be the smallest positive rational number such that the character $q_m \cdot \chi_m$ lifts to a character of the group P_m.

Lemma 2.64. *The conjugate class of the parabolic subgroup P_m only depends on the rank of m.*

Proof. Given m, $m' \in \mathfrak{m}^+$ of the same rank, there exists some $h \in H^{\mathbb{C}}$ such that $m' = \mathrm{Ad}(h)m$. We have that

$$\mathrm{Ker}(\mathrm{ad}(m')_{|\mathfrak{h}^{\mathbb{C}}}) = \mathrm{Ad}(h)\,\mathrm{Ker}(\mathrm{ad}(m)_{|\mathfrak{h}^{\mathbb{C}}}) \quad \mathrm{Im}(\mathrm{ad}(m')_{|\mathfrak{m}^-}) = \mathrm{Ad}(h)\,\mathrm{Im}(\mathrm{ad}(m)_{|\mathfrak{m}_-}),$$

i.e., $\mathfrak{p}_m = \mathrm{Ad}(h)\mathfrak{p}_{m'}$, and therefore $P_m = hP_{m'}h^{-1}$. ☐

Remark 2.65. For the parabolic subgroups and algebras we use the following notation

- P_m, $\mathfrak{p}_m$ for $m \in \mathfrak{m}^+$, as defined in Section 2.5.1
- P_s, $\mathfrak{p}_s$ for $s \in i\mathfrak{h}$, as defined in Section 2.5.2.
- $P_{r'}$, $\mathfrak{p}_{r'}$ for $1 \leq r' \leq r$, referring to P_m and $\mathfrak{p}_m$ for m of rank r', by Lemma 2.64.

Lemma 2.66. *Let $H'^{\mathbb{C}}$ be the stabilizer of m in $H^{\mathbb{C}}$ via the action $H^{\mathbb{C}} \to \mathrm{Aut}(\mathfrak{m}^{\mathbb{C}})$ and $\mathfrak{h}'^{\mathbb{C}}$ be the annihilator of m in $\mathfrak{h}^{\mathbb{C}}$ via the action $\mathfrak{h}^{\mathbb{C}} \to \mathrm{End}(\mathfrak{m}^{\mathbb{C}})$. Let $m \in \mathfrak{m}^+$ be an element of maximal rank. The parabolic subgroups of $H'^{\mathbb{C}}$ are in correspondence with the parabolic subgroups of $H^{\mathbb{C}}$ given by $s \in i\mathfrak{h}$ such that $m \in \mathfrak{m}_s$.*

Proof. An element $s \in i\mathfrak{h}'$ defines a parabolic subgroup P'_s. If we consider s as an element of $i\mathfrak{h}$, it defines a parabolic subgroup P_s of $H^{\mathbb{C}}$. Since $\mathfrak{h}'$ annihilates m, we have $m \in \mathfrak{m}_s^0 \subset \mathfrak{m}_s$. Conversely, if $s \in i\mathfrak{h}$ is such that $m \in \mathfrak{m}_s^0$, then s annihilates m, i.e., $s \in i\mathfrak{h}'$ and determines a parabolic subgroup P'_s of $H'^{\mathbb{C}}$. ☐

2.5.3 Subtubes and Levi factors

As in the previous section, we begin by considering the case of a fixed subset Γ' of a system of st-orthogonal roots Γ.

Define $\mathfrak{t}^{\mathbb{C}}_{\Gamma'} = \langle h_\gamma \mid \gamma \in Q_{<<} \rangle \subset \mathfrak{t}^{\mathbb{C}}$ and consider the following vector subspaces or subalgebras

$$\begin{aligned} \mathfrak{m}^{\pm}_{\Gamma'} &= \mathfrak{g}^{\mathbb{C}}[\pm Q_{<<} \cup \Gamma'] \subset \mathfrak{m}^{\pm}, & \mathfrak{m}^{\mathbb{C}}_{\Gamma'} &= \mathfrak{m}^{+}_{\Gamma'} + \mathfrak{m}^{-}_{\Gamma'} \subset \mathfrak{m}^{\mathbb{C}}, \\ \mathfrak{h}^{\mathbb{C}}_{\Gamma'} &= \mathfrak{t}^{\mathbb{C}}_{\Gamma'} \cup \mathfrak{g}^{\mathbb{C}}[C_{<<}] \subset \mathfrak{h}^{\mathbb{C}}, & \mathfrak{g}^{\mathbb{C}}_{\Gamma'} &= \mathfrak{h}^{\mathbb{C}}_{\Gamma'} + \mathfrak{m}^{\mathbb{C}}_{\Gamma'} \subset \mathfrak{g}^{\mathbb{C}}. \end{aligned}$$

as well as their real forms

$$\mathfrak{m}_{\Gamma'} = \mathfrak{m} \cap \mathfrak{m}^{\mathbb{C}}_{\Gamma'} \subset \mathfrak{m}, \qquad \mathfrak{h}_{\Gamma'} = \mathfrak{h} \cap \mathfrak{h}^{\mathbb{C}}_{\Gamma'} \subset \mathfrak{h}, \qquad \mathfrak{g}_{\Gamma'} = \mathfrak{g} \cap \mathfrak{g}^{\mathbb{C}}_{\Gamma'} \subset \mathfrak{g}.$$

Example 2.67. *For $\mathfrak{g} = \mathfrak{su}(p, q)$ we have*

$$\left(\begin{array}{cc|ccc}
0 & & & & \\
 & \mathfrak{h}^{\mathbb{C}}_{\Gamma'} & \mathfrak{m}^-_{\Gamma'} & & \\ \hline
 & \mathfrak{m}^+_{\Gamma'} & \mathfrak{h}^{\mathbb{C}}_{\Gamma'} & & \\
 & & & 0 & \\
 & & & & 0
\end{array}\right).$$

Corresponding to $\mathfrak{h}_{\Gamma'}$ and $\mathfrak{g}_{\Gamma'}$ there are subgroups $G_{\Gamma'} \subset G$ and $H_{\Gamma'} \subset H$, in such a way that $(G_{\Gamma'}, H_{\Gamma'})$ is a symmetric pair of tube type with Cartan decomposition $\mathfrak{g}_{\Gamma'} = \mathfrak{h}_{\Gamma'} + \mathfrak{m}_{\Gamma'}$. The set Γ' is now seen as a system of st-orthogonal roots inside $\Delta(\mathfrak{g}^{\mathbb{C}}_{\Gamma'}, \mathfrak{t}^{\mathbb{C}}_{\Gamma'})$. Thus, the Toledo character with respect to it is $\chi_{\Gamma'} = \sum_{\gamma \in \Gamma'} \gamma$. Since it is a tube space, $\mathfrak{m}^+_{\Gamma'}$ can be endowed with the structure of Jordan algebra. Let $\det_{\Gamma'} : \mathfrak{m}^+_{\Gamma'} \to \mathbb{C}$ be its determinant, which is a polynomial of degree r'.

We apply Lemma 2.46 to get the following lemma.

Lemma 2.68. *There exists $q_{\Gamma'} \in \mathbb{Q}$ such that the character $q_{\Gamma'} \cdot \chi_{\Gamma'}$ lifts to a character $\tilde{\chi}_{\Gamma'}$ of the group $H^{\mathbb{C}}_{\Gamma'}$. Let q be a positive integer multiple of $q_{\Gamma'}$. For $h \in H^{\mathbb{C}}_{\Gamma'}$ and $X \in \mathfrak{m}^+_{\Gamma'}$ we have that*

$$\det_{\Gamma'}(h \cdot X)^q = (\tilde{\chi}_{\Gamma'}(h))^{q/q_{\Gamma'}} \det_{\Gamma'}(X)^q.$$

We return now to the parabolic subgroup $P_{e_{\Gamma'}}$, which we write as $P'_s = P_{e_{\Gamma'}}$ for some $s' \in i\mathfrak{h}$. We have

$$\mathfrak{p}_{s'} = \mathfrak{g}^{\mathbb{C}}[C_{>>}, C_{<<}, C_{<>}, C_{>}, \pm C_{<}, C_0], \qquad \mathfrak{m}^+_{\Gamma'} = \mathfrak{g}^{\mathbb{C}}[Q_{<<}].$$

The subalgebra $\mathfrak{p}_{s'}$ and the corresponding group $P_{s'}$ act on $\mathfrak{m}^+_{\Gamma'}$ via the adjoint action. The unipotent radical of $\mathfrak{p}_{s'}$ is $\mathfrak{u}_{s'} = \mathfrak{g}^{\mathbb{C}}[C_{<>}, C_{<}]$, which acts trivially on $\mathfrak{m}^+_{\Gamma'}$. Therefore the action descends to the Levi factor $L_{s'}$ where $P_{s'} \to L_{s'}$ is a projection with kernel the unipotent radical.

Consider the kernel of the representation $\mathrm{Ad} : L_{s'} \to \mathrm{Aut}(\mathfrak{m}^+_{\Gamma'})$, given by

$$\mathrm{Ker}_{\mathrm{Ad}}(L_{s'}) = \{g \in L_{s'} \mid \mathrm{Ad}(g) = \mathrm{Id}\}.$$

The action clearly factors again through $\widetilde{L_{s'}} := L_{s'} / \mathrm{Ker}_{\mathrm{Ad}}(L_{s'})$:

$$\begin{tikzcd}
P_{s'} \arrow[r] \arrow[drr] & L_{s'} \arrow[r] \arrow[dr] & \widetilde{L_{s'}} \arrow[d] \\
 & & \mathrm{Aut}(\mathfrak{m}^+_{\Gamma'}).
\end{tikzcd} \tag{2.68.1}$$

The Lie algebra of $\widetilde{L_{s'}}$ is $\widetilde{\mathfrak{l}_{s'}} = \{X + \mathfrak{u} \mid X \in \mathfrak{g}^{\mathbb{C}}[C_{<<}]\} \subset \mathfrak{l}_{s'}/\mathfrak{u}_{s'}$. This gives an isomorphism with $\mathfrak{h}^{\mathbb{C}}_{\Gamma'}$ that lifts to the corresponding Lie group, for which we can write a version of Lemma 2.68.

Lemma 2.69. *There is an isomorphism $\psi : \widetilde{L_{s'}} \to H_{\Gamma'}$, in such a way that*

$$\mathrm{Ad}(h)X = \mathrm{Ad}(\psi(h))X \text{ for } X \in \mathfrak{m}_{\Gamma'}^{+}$$

and therefore a character $\chi_{\tilde{L}}$ can be defined in the Lie group $\widetilde{L_{s'}}$. For q a positive integer multiple of $q_{\Gamma'}$, this character lifts to a character which we call again $(\tilde{\chi}_{s'}(h))^{q/q_{\Gamma'}}$ of the group $\widetilde{L_{s'}}$. Moreover, for $h \in \widetilde{L_{s'}}$ and $X \in \mathfrak{m}_{\Gamma'}^{+}$ this character satisfies

$$\det{}_{\Gamma'}(h \cdot X)^q = (\tilde{\chi}_{s'}(h))^{q/q_{\Gamma'}} \det{}_{\Gamma'}(X)^q. \tag{2.69.1}$$

Now let $m \in \mathfrak{m}^+$ be of maximal rank. We finish this section by establishing a relation between the normalizer of a subtube and a Levi subgroup of the parabolic subgroup P_m. As above, for Γ a system of st-orthogonal roots take $m = e_\Gamma = e_{\gamma_1} + \ldots + e_{\gamma_r}$. The system Γ defines the Cayley transform and hence, the subtube algebras $\mathfrak{g}_T$, $\mathfrak{h}_T$ and groups G_T, H_T.

We make use of restricted roots. In the maximal case, the preceding diagram becomes

$$\left(\begin{array}{c|cc} C_{<<} & Q_{<<} & Q_{<} \\ \hline -Q_{<<} & C_{<<} & C_{<} \\ -Q_{<} & -C_{<} & C_0 \end{array}\right).$$

Consider the parabolic subgroup P_{e_Γ} and let $L_{e_\Gamma} = L_{s_\chi}$ be the Levi subgroup defined in Section 2.5.1.

Lemma 2.70. *We have that $N_{H^{\mathbb{C}}}(\mathfrak{h}_T^{\mathbb{C}})_0$ is isomorphic to L_{e_Γ} and is therefore a Levi subgroup $L \subset H^{\mathbb{C}}$.*

Proof. We first prove that $\mathfrak{n} := \mathfrak{n}_{\mathfrak{h}^{\mathbb{C}}}(\mathfrak{h}_T^{\mathbb{C}})$ is equal to $\mathfrak{l} \subset \mathfrak{h}^{\mathbb{C}}$. We have that

$$\mathfrak{n} = \{X \in \mathfrak{h}^{\mathbb{C}} \mid \mathrm{ad}(X)(\mathfrak{h}_T^{\mathbb{C}}) \subset \mathfrak{h}_T^{\mathbb{C}}\}$$

$$\mathfrak{l} = \{X \in \mathfrak{h}^{\mathbb{C}} \mid \mathrm{ad}(X)(s_\chi) = 0\}.$$

An element $X \in \mathfrak{h}^{\mathbb{C}}$ normalizes $\mathfrak{h}_T^{\mathbb{C}}$ if and only if it normalizes all the root vectors e_α for $\alpha \in C_{<<}$. By restricted root theory we have that e_β belongs to $\mathfrak{n}$ if and only if $\beta \in C_{<<} \cup C_0$. By Lemma 2.30, for $\beta \in C_j$, there exists $\alpha \in C_{ij}$ such that $[e_\alpha, e_\beta] = e_{\alpha+\beta} \in C_i$. It only remains to show if a linear combination of elements of $\mathfrak{g}^{\mathbb{C}}[\pm C_<]$ can normalize $\mathfrak{h}_T^{\mathbb{C}}$. This is not possible, as there will be a root $\beta \in C_{ij}$ such that $[\sum_j e_{\alpha_j}, e_\beta]$ is a sum of root vectors with roots projecting to $\frac{1}{2}\gamma_j$ or $-\frac{1}{2}\gamma_i$. As the roots are different, this sum is not zero, and does not belong to $\mathfrak{h}_T^{\mathbb{C}}$. Therefore, $\mathfrak{n} = \mathfrak{g}^{\mathbb{C}}[C_{<<}, C_0]$.

On the other hand, for $\mathfrak{l}$, we have that $s_\chi = s_{\chi_T} - s_{\chi_\Gamma}$. We know that χ_T commutes with $\mathfrak{h}^{\mathbb{C}}$, so the whole $\mathfrak{h}^{\mathbb{C}}$ annihilates s_{χ_T}. The character χ_Γ equals $\sum_{\gamma\in\Gamma}\gamma$, so $s_{\chi_\Gamma} = \frac{2}{B(\gamma_1,\gamma_1)}\sum_{\gamma\in\Gamma} h_\gamma$, since all have the same norm. Similarly to $\mathfrak{n}$, we have that $e_\alpha \in \mathfrak{l}$ for $\alpha \in C_{<<}, C_0$, and $e_\alpha \notin \mathfrak{l}$ for $\alpha \in \pm C_<$. As before, no linear combination of root vectors can annihilate s_χ.

As their Lie algebras coincide and both groups are connected, they coincide. □

Lemma 2.71. *For γ of maximal rank, the subalgebra $\mathfrak{m}^0_\chi$ coincides with $\mathfrak{m}^{\mathbb{C}}_T$.*

Proof. By the proof of Proposition 2.60, we know that $\mathfrak{m}^0_\chi = \mathfrak{g}^{\mathbb{C}}[\pm Q_{<<}]$. In general, $\mathfrak{m}^{\mathbb{C}}_T = \mathfrak{g}^{\mathbb{C}}[\pm Q_{<<}, \pm Q_{<>}, \pm Q_{>>}]$, but as we are in the maximal case, $\pm Q_{<>}$ and $\pm Q_{>>}$ are empty and $\mathfrak{m}^{\mathbb{C}}_T = \mathfrak{m}^0_\chi$. □

2.6 Normalizer of the maximal tube subdomain

Let G be a simple Hermitian group of non-tube type with maximal compact subgroup H. In this section we establish some relations between some subgroups of the group $H^{\mathbb{C}}$. They will be relevant for the study of the moduli space of G-Higgs bundle with maximal Toledo invariant.

Let $G_T \subset G$, be the maximal imbedded subgroups of tube type inside G and $H_T \subset H$, its corresponding maximal compact subgroup, i.e. G_T/H_T is a tube-type space, maximal among the tube-type subspaces of G/H. These can be determined by giving a system of st-orthogonal roots, or alternatively, an element of maximal rank in $\mathfrak{m}^-$ or $\mathfrak{m}^+$. Let N be the identity component of the normalizer of $H^{\mathbb{C}}_T$ into $H^{\mathbb{C}}$, C_G be the centralizer of $\mathfrak{g}^{\mathbb{C}}_T$ in $H^{\mathbb{C}}$ and C^{ss}_G be its semisimple part. The aim of this section is to show that there exist an exact sequence

$$N \to \frac{N}{H^{\mathbb{C}}_T} \times \frac{N}{C^{ss}_G} \to Q \to 1,$$

with Q a 1-dim group. Moreover, under some hypothesis we can show that the first map is injective and the group N injects.

The proofs of the stated results are independent of the classification theorem of Lie groups. However, in order to illustrate them, we put at the right-hand side the example of $\mathrm{SU}(p,q)$ in every step.

First, consider N to be the identity component of the normalizer of $H^{\mathbb{C}}_T$ into $H^{\mathbb{C}}$ and C to be the centralizer C of $H^{\mathbb{C}}_T$ in $H^{\mathbb{C}}$.

$$H_T^{\mathbb{C}} \qquad \left\{ \left(\begin{array}{c|cc} A & & \\ \hline & B & \\ & & 1 \end{array}\right) \mid \det A \det B = 1 \right\} \subset \mathrm{SU}(p,q),$$

$$N := (N_{H^{\mathbb{C}}}(H_T^{\mathbb{C}}))_0 \qquad \left\{ \left(\begin{array}{c|cc} A & & \\ \hline & B & \\ & & D \end{array}\right) \mid \det A \det B \det D = 1 \right\} \subset \mathrm{SU}(p,q),$$

$$C := C_{H^{\mathbb{C}}}(H_T^{\mathbb{C}}) \qquad \left\{ \left(\begin{array}{c|cc} \lambda \,\mathrm{Id} & & \\ \hline & \mu \,\mathrm{Id} & \\ & & D \end{array}\right) \mid \lambda^p \mu^p \det D = 1 \right\} \subset \mathrm{SU}(p,q).$$

From [KW65], using the notation $\widetilde{\mathfrak{h}}_T^{\mathbb{C}} = \mathfrak{n}_{\mathfrak{h}^{\mathbb{C}}}(\mathfrak{h}_T^{\mathbb{C}})$, we have that $\widetilde{\mathfrak{h}}_T^{\mathbb{C}} = \mathfrak{h}_T^{\mathbb{C}} + \mathfrak{z}_{\mathfrak{h}^{\mathbb{C}}}(\mathfrak{h}_T^{\mathbb{C}})$, where $\mathfrak{z}_{\mathfrak{h}^{\mathbb{C}}}(\mathfrak{h}_T^{\mathbb{C}})$ denotes the centralizer of $\mathfrak{h}_T^{\mathbb{C}}$ in $\mathfrak{h}^{\mathbb{C}}$. We then have $N = CH_T^{\mathbb{C}}$, and by the second isomorphism theorem, we obtain $\frac{N}{H_T^{\mathbb{C}}} \cong \frac{C}{Z}$, where $Z = H_T^{\mathbb{C}} \cap C_{H^{\mathbb{C}}}(H_T^{\mathbb{C}}) = Z(H^{\mathbb{C}})$.

In the case of $\mathrm{SU}(p,q)$, the equality $CH_T^{\mathbb{C}} = N$ is given by

$$\left(\begin{array}{c|cc} \lambda \,\mathrm{Id} & & \\ \hline & \mu \,\mathrm{Id} & \\ & & C \end{array}\right) \cdot \left(\begin{array}{c|cc} A & & \\ \hline & B & \\ & & 1 \end{array}\right) = \left(\begin{array}{c|cc} \lambda A & & \\ \hline & \mu B & \\ & & C \end{array}\right).$$

The intersection between $H_T^{\mathbb{C}}$ and $C_{H^{\mathbb{C}}}(H_T^{\mathbb{C}})$ is

$$Z = Z(H^{\mathbb{C}}) \qquad \left\{ \left(\begin{array}{c|cc} \lambda \,\mathrm{Id} & & \\ \hline & \mu \,\mathrm{Id} & \\ & & \mathrm{Id} \end{array}\right) \mid \lambda^p \mu^p = 1 \right\} \subset \mathrm{SU}(p,q).$$

Define now the group

$$C_G := C_{H^{\mathbb{C}}}(\mathfrak{g}_T^{\mathbb{C}}) \qquad \left\{ \left(\begin{array}{c|cc} \lambda \,\mathrm{Id} & & \\ \hline & \lambda \,\mathrm{Id} & \\ & & C \end{array}\right) \mid \lambda^{2p} \det C = 1 \right\} \subset \mathrm{SU}(p,q),$$

which in the case of $\mathrm{SU}(p,q)$ is given by the elements of $C_{H^{\mathbb{C}}}(H_T^{\mathbb{C}})$ such that $\lambda = \mu$. Take the semisimple part C_G' and the centre $Z(C_G)$ of C_G,

$$C_G' = [C_G, C_G] \qquad \left\{ \left(\begin{array}{c|cc} \mathrm{Id} & & \\ \hline & \mathrm{Id} & \\ & & C \end{array}\right) \mid \det C = 1 \right\} \subset \mathrm{SU}(p,q),$$

$$Z(C_G) \qquad \left\{ \left(\begin{array}{c|cc} \lambda \,\mathrm{Id} & & \\ \hline & \lambda \,\mathrm{Id} & \\ & & \nu \,\mathrm{Id} \end{array}\right) \mid \lambda^{2p} \nu^{q-p} = 1 \right\} \subset \mathrm{SU}(p,q).$$

Lemma 2.72. *We have that* $N = H_T^{\mathbb{C}} C_G$, $H_T^{\mathbb{C}} \cap C_G = Z(H_T^{\mathbb{C}}) \cap Z(C_G)$.

Proof. Using the notation and arguments similar to those of Sections 2.5.2 and 2.5.3, we have that $\mathfrak{n} = \mathfrak{t}^{\mathbb{C}} \cup \mathfrak{g}^{\mathbb{C}}[C_{<<}, C_0]$, where $\mathfrak{t}^{\mathbb{C}} \subset \mathfrak{h}^{\mathbb{C}}$ is a Cartan subalgebra of $\mathfrak{g}^{\mathbb{C}}$, i.e., a maximal abelian subalgebra. We have that $\mathfrak{t}_\Gamma^{\mathbb{C}} = \langle h_\gamma \mid \gamma \in \Gamma \rangle$ is contained in $\mathfrak{t}^{\mathbb{C}}$ and it is a maximal abelian subalgebra of $\mathfrak{g}_T^{\mathbb{C}}$. We define a complement of $\mathfrak{t}_\Gamma^{\mathbb{C}}$ in $\mathfrak{t}^{\mathbb{C}}$ by taking $\mathfrak{t}_c^{\mathbb{C}} = \mathfrak{t}^{\mathbb{C}} \cap \mathfrak{c}_{\mathfrak{h}^{\mathbb{C}}}(\mathfrak{g}_T^{\mathbb{C}})$, maximal abelian subalgebra of $\mathfrak{c}_{\mathfrak{h}^c}(\mathfrak{g}_T^{\mathbb{C}})$. Therefore,

$$\mathfrak{t}^{\mathbb{C}} = \mathfrak{t}_\Gamma^{\mathbb{C}} \cup \mathfrak{g}_c^{\mathbb{C}}, \qquad \mathfrak{h}_T^{\mathbb{C}} = \mathfrak{t}_\Gamma^{\mathbb{C}} \cup \mathfrak{g}^{\mathbb{C}}[C_{<<}], \qquad \mathfrak{c}_{\mathfrak{h}^{\mathbb{C}}}(\mathfrak{g}_T^{\mathbb{C}}) = \mathfrak{t}_c^{\mathbb{C}} \cup \mathfrak{g}^{\mathbb{C}}[C_0],$$

so we have $\mathfrak{n} = \mathfrak{h}_T^{\mathbb{C}} \oplus \mathfrak{c}_{\mathfrak{h}^{\mathbb{C}}}(\mathfrak{g}_T^{\mathbb{C}})$. Since N is connected, $N = H_T^{\mathbb{C}} C_G$. For the second equality, an element in $H_T^{\mathbb{C}}$ commutes with C_G, as C_G is contained in $C_{H^{\mathbb{C}}}(H_T^{\mathbb{C}})$. If this element is also in C_G, it then belongs to the centre $Z(C_G)$. An element of C_G commutes with $H_T^{\mathbb{C}}$. If this element lies in $H_T^{\mathbb{C}}$, it then belongs to the centre $Z(H_T^{\mathbb{C}})$. Conversely, $Z(H_T^{\mathbb{C}}) \cap Z(C_G)$ trivially belongs to $H_T^{\mathbb{C}} \cap C_G$. □

As a direct consequence,

$$N = H_T^{\mathbb{C}} C_G' Z(C_G), \tag{2.72.1}$$

and by the second isomorphism theorem we have,

$$\frac{N}{H_T^{\mathbb{C}} C_G'} \cong \frac{Z(C_G)}{H_T^{\mathbb{C}} C_G' \cap Z(C_G)}.$$

Define the groups

$$\Gamma_H := H_T^{\mathbb{C}} \cap Z(C_G) \qquad \left\{ \left(\begin{array}{c|cc} \lambda\,\mathrm{Id} & & \\ \hline & \lambda\,\mathrm{Id} & \\ & & \mathrm{Id} \end{array} \right) \mid \lambda^{2p} = 1 \right\},$$

$$\Gamma_C := C_G' \cap Z(C_G) \qquad \left\{ \left(\begin{array}{c|cc} \mathrm{Id} & & \\ \hline & \mathrm{Id} & \\ & & \nu\,\mathrm{Id} \end{array} \right) \mid \nu^{q-p} = 1 \right\}.$$

From Lemma 2.72, we have that $H_T^{\mathbb{C}} \cap C_G' = \Gamma_H \cap \Gamma_C$.

Remark 2.73. When the complexification of G exists, the group C_G equals $C_{G^{\mathbb{C}}}(G_T^{\mathbb{C}})$ and $\Gamma = Z(G_T^{\mathbb{C}})$.

Lemma 2.74. *The group* $H_T^{\mathbb{C}} C_G' \cap Z(C_G)$ *is isomorphic to* $\Gamma_H \Gamma_C$.

Proof. On the one hand, we have the trivial inclussion

$$\Gamma_H \Gamma_C = (Z(C_G) \cap H_T^{\mathbb{C}})(Z(C_G) \cap C'_G) \subset H_T^{\mathbb{C}} C'_G \cap Z(C_G).$$

On the other hand, one has

$$H_T^{\mathbb{C}} C'_G \cap Z(C_G) \subset (H_T^{\mathbb{C}} \cap Z(C_G) C'_G)(C'_G \cap Z(C_G) H_T^{\mathbb{C}}),$$

as $hc = z$ with $h \in H_T^{\mathbb{C}}$, $c \in C'_G$ and $z \in Z(C_G)$ gives $h = zc^{-1} \in Z(C_G)C'_G$ and $c = zh^{-1} \in Z(C_G)H_T^{\mathbb{C}}$. Moreover,

$$H_T^{\mathbb{C}} C'_G \cap Z(C_G) \subset (H_T^{\mathbb{C}} \cap Z(C_G)C'_G)(C'_G \cap Z(C_G)H_T^{\mathbb{C}}) = \Gamma(H_T^{\mathbb{C}} \cap C'_G)\Gamma'(H_T^{\mathbb{C}} \cap C'_G) \subset \Gamma_H \Gamma_C,$$

since $H_T^{\mathbb{C}} \cap C'_G \subset Z(C_G)$ gives $H_T^{\mathbb{C}} \cap C'_G \subset H_T^{\mathbb{C}} \cap Z(C_G) = \Gamma_H$ or equivalently $H_T^{\mathbb{C}} \cap C'_G \subset Z(C_G) \cap C'_G = \Gamma_C$ □

Consider the map $N \to \frac{N}{H_T^{\mathbb{C}}} \times \frac{N}{C'_G}$ given by $n \mapsto (nH_T^{\mathbb{C}}, nC'_G)$ and define

$$Q = \frac{\frac{N}{H_T^{\mathbb{C}}} \times \frac{N}{C'_G}}{\text{im } N}.$$

Proposition 2.75. *We have that*

$$Q \cong \frac{N}{H_T^{\mathbb{C}} C'_G} \cong \frac{Z(C_G)}{\Gamma\Gamma'}.$$

Proof. The map $nH_T^{\mathbb{C}}C'_G \mapsto (eH_T^{\mathbb{C}}, nC'_G)\text{im }(N) = (nH_T^{\mathbb{C}}, eC'_G)\text{im } N$ defines the isomorphism.

The map $\frac{N}{H_T^{\mathbb{C}}} \times \frac{N}{C'_G} \to \frac{Z(C_G)}{\Gamma\Gamma'}$ in the first sequence is defined as follows. Given $(n_1 H_T^{\mathbb{C}}, n_2 C'_G)$, by the decomposition $N = H_T^{\mathbb{C}} C'_G Z(C_G)$ there exist $h_i \in H_T^{\mathbb{C}}$, $c_i \in C'_G$ and $z_i \in Z(C_G)$ such that $n_i = h_i c_i z_i$. We define the image of $(n_1 H_T^{\mathbb{C}}, n_2 C'_G)$ as $c_1^{-1} c_2 \Gamma_H \Gamma_C$. □

As a consequence of Proposition 2.75, when $C'_G \cap H_T^{\mathbb{C}} = 1$, we have an exact sequence

$$1 \to N \to \frac{N}{H_T^{\mathbb{C}}} \times \frac{N}{C'_G} \to \frac{Z(C_G)}{\Gamma\Gamma'} \to 1. \tag{2.75.1}$$

Remark 2.76. Consider the group

$$C_{H^{\mathbb{C}}}(C_G) \qquad \left\{ \left(\begin{array}{c|cc} A & & \\ \hline & B & \\ & & \nu\,\text{Id} \end{array} \right) \mid \det A \det B \nu^{q-p} = 1 \right\} \subset \text{SU}(p,q).$$

We also have isomorphisms

$$\frac{N}{H_T^{\mathbb{C}}} \cong \frac{C_G}{\Gamma_H} \qquad\qquad \frac{N}{C'_G} \cong \frac{C_{H^{\mathbb{C}}}(C_G)}{\Gamma_C}.$$

Example 2.77. *For* $\mathrm{SU}(p,q)$ *the injection* $N \to \frac{N}{H_T^{\mathbb{C}}} \times \frac{N}{C_G'}$*, together with these isomorphisms accounts for the mapping*

$$\left(\begin{array}{c|cc} A & & \\ \hline & B & \\ & & D \end{array}\right) \mapsto \left(D \quad , \quad \left(\begin{array}{c|cc} A & & \\ \hline & B & \\ & & 1 \end{array}\right)\right).$$

Note that $\det A \det B \det D = 1$*, while* $\det D$ *and* $\det A \det B$ *are not necessarily one.*

Lemma 2.78. *Assume that* Γ_H *is finite and let* $o(\Gamma_H)$ *be the maximum of the orders of the elements of the finite group* Γ_H*. There is a map* $Q = \frac{Z(C_G)}{\Gamma_H \Gamma_C} \to \frac{N}{C_G'}$*, so that the map* $Q \to \frac{N}{C_G'} \to \frac{N}{H_T^{\mathbb{C}}} \times \frac{N}{C_G'} \to Q$ *is the* $o(\Gamma_H)$*-power.*

Proof. We define the map by $z\Gamma_H\Gamma_C \mapsto z^{o(\Gamma_H)}C_G'$, for $z \in Z(C_G)$. This is well-defined because $\Gamma_C \subset C_G'$ and any element Γ_H to the power of $o(\Gamma_H)$ is the identity. Trivially, the composition is the $o(\Gamma_H)$-power. □

Remark 2.79. We believe that the hypothesis $C_G' \cap H_T^{\mathbb{C}} = 1$ in proposition 2.75 is always satisfied, but we do not have a classification independent proof for this fact. See Remark B.5 in the appendix.

Chapter 3

Higgs bundles

3.1 Basics on twisted Higgs bundles and stability

Let X be a Riemann surface of genus g. We define the notion of G-Higgs bundle on X for any real reductive Lie group G, not necessarily Hermitian. Let H be a maximal compact subgroup of G. The Lie algebra of $\mathfrak{g}$ has a decomposition $\mathfrak{g} = \mathfrak{h} + \mathfrak{m}$, where $\mathfrak{h}$ is the Lie algebra of H and $\mathfrak{m}$ is a complementary vector subspace, orthogonal to $\mathfrak{h}$ with respect to a given metric in $\mathfrak{g}$. This decomposition generalizes the Cartan decomposition for semisimple Lie algebras, where the Killing form can be taken as metric, since it satisfies $[\mathfrak{m}, \mathfrak{m}] \subset \mathfrak{h}$, $[\mathfrak{m}, \mathfrak{h}] \subset \mathfrak{h}$. From the isotropy representation $H \to \mathrm{Aut}(\mathfrak{m})$, we get the representation $\iota : H^{\mathbb{C}} \to \mathrm{Aut}(\mathfrak{m}^{\mathbb{C}})$. Denote by K the canonical bundle over the surface X, which equals $T^*X^{(1,0)}$. **A G-Higgs bundle on** X consists of a holomorphic principal $H^{\mathbb{C}}$-bundle E together with a holomorphic section $\varphi \in H^0(E(\mathfrak{m}^{\mathbb{C}}) \otimes K)$, where $E(\mathfrak{m}^{\mathbb{C}})$ is the associated vector bundle with fibre $\mathfrak{m}^{\mathbb{C}}$ via the complexified isotropy representation.

Example 3.1. *Let G be a compact group. Then we have $H = G$ and $\mathfrak{m} = 0$. A G-Higgs bundle is a principal $G^{\mathbb{C}}$-bundle with a zero Higgs field, i.e., a holomorphic principal $G^{\mathbb{C}}$-bundle.*

Example 3.2. *Let H be a compact group and consider $G = H^{\mathbb{C}}$. We have that H is a maximal compact subgroup of G, and $\mathfrak{m} = i\mathfrak{h}$. In this case, a G-Higgs bundle is a principal $H^{\mathbb{C}}$-bundle together with a section $\varphi \in H^0(E(\mathfrak{h}^{\mathbb{C}}) \otimes K) = H^0(E(\mathfrak{g}) \otimes K)$.*

Definition 3.3. *Let G' be a reductive subgroup of G. A maximal compact subgroup of G' is given by $H' = H \cap G'$ and we can take a compatible Cartan decomposition, in the sense that $\mathfrak{h}' \subset \mathfrak{h}$ and $\mathfrak{m}' \subset \mathfrak{m}$. Moreover, the isotropy representation of H' is the restriction of the isotropy representation of H. We say that the structure group* *of a G-Higgs bundle (E,φ) reduces to G' when there is a reduction of the structure group of the underlying $H^{\mathbb{C}}$-bundle to $H'^{\mathbb{C}}$, given by a subbundle E_σ, and the Higgs field $\varphi \in H^0(E(\mathfrak{m}^{\mathbb{C}})\otimes K)$ belongs to $H^0(E_\sigma(\mathfrak{m}'^{\mathbb{C}})\otimes K)$.*

We borrow the notion of parameter-depending stability for G-Higgs bundles from [GGM12] (an updated version of [GGM09]). Recall from Section 2.5.2 that an element $s \in i\mathfrak{h}$ determines an R-parabolic subgroup P_s together with an antidominant character χ_s of $\mathfrak{p}_s$. For $\sigma \in \Gamma(E(H^{\mathbb{C}}/P_s))$ a reduction of the structure group of E from $H^{\mathbb{C}}$ to P_s, we define the degree relative to σ and s, or equivalently to σ and χ_s, as follows. When we have that a real multiple $\mu\chi_s$ of the character exponentiates to a character $\tilde{\chi}_s$ of P_s, we compute the degree as

$$\deg(E)(\sigma,s) = \deg(E)(\sigma,\chi_s) = \frac{1}{\mu}\deg(E_\sigma(\tilde{\chi}_s)).$$

This condition is not always satisfied, but one shows (Section 4.6 in [GGM09]) that the antidominant character can be expressed as a linear combination of characters of the centre and fundamental weights, $\chi_s = \sum_j z_j\mu_j + \sum_k n_k\lambda_k$. Lemma 2.4 in [GGM09] states that there exists an integer multiple m of the characters of the centre and the fundamental weights exponentiating to the group, so we can define the degree as

$$\deg(E)(\sigma,s) = \deg(E)(\sigma,\chi_s) = \frac{1}{m}\left(\sum_j z_j \deg(E_\sigma(\widetilde{m\mu_j})) + \sum_k n_k \deg(E_\sigma(\widetilde{m\lambda_k}))\right).$$

This value is independent of the expression of χ_s as sum of characters and the integer n.

We define the subalgebra $i\mathfrak{h}_\iota$ as follows. Consider the decomposition $\mathfrak{h} = \mathfrak{z} + [\mathfrak{h},\mathfrak{h}]$, where $\mathfrak{z}$ is the centre of $\mathfrak{h}$, and the isotropy representation $d\iota = \mathrm{ad} : \mathfrak{h} \to \mathrm{End}(\mathfrak{m})$. Let $\mathfrak{z}' = \ker(d\iota_{|\mathfrak{z}})$ and take $\mathfrak{z}''$ such that $\mathfrak{z} = \mathfrak{z}' + \mathfrak{z}''$. Define the subalgebra $\mathfrak{h}_\iota := \mathfrak{z}'' + [\mathfrak{h},\mathfrak{h}]$. The subindex ι denotes that we have taken away the part of the centre $\mathfrak{z}$ acting trivially via the isotropy representation $d\iota$.

Remark 3.4. For groups of Hermitian type, $\mathfrak{z}' = 0$ since an element both in $\mathfrak{z}(\mathfrak{h})$ and $\ker(d\iota)$ belongs to the centre of $\mathfrak{g}$, which is zero, as $\mathfrak{g}$ is semisimple. Hence $\mathfrak{h}_\iota = \mathfrak{h}$.

We recall the definitions of $\mathfrak{m}_s$ and $\mathfrak{m}_s^0$,

$$\begin{aligned}\mathfrak{m}_s &= \{Y \in \mathfrak{m}^{\mathbb{C}} \ :\ \mathrm{Ad}(e^{ts})Y \text{ is bounded as } t \to \infty\}\\ \mathfrak{m}_s^0 &= \{Y \in \mathfrak{m}^{\mathbb{C}} \ :\ \mathrm{Ad}(e^{ts})Y = Y \ \text{ for every } t\},\end{aligned}$$

as given in Section 2.5.1.

Definition 3.5. *Let $\alpha \in i\mathfrak{z} \subset \mathfrak{z}^{\mathbb{C}}$. We say that a G-Higgs bundle (E, φ) is:*

- α-**semistable** *if for any $s \in i\mathfrak{h}$ and any holomorphic reduction $\sigma \in \Gamma(E(H^{\mathbb{C}}/P_s))$ such that $\varphi \in H^0(E_\sigma(\mathfrak{m}_s) \otimes K)$, we have that* $\deg(E)(\sigma, s) - \langle \alpha, s \rangle \geq 0$.
- α-**stable** *if for any $s \in i\mathfrak{h}_\iota$ and any holomorphic reduction $\sigma \in \Gamma(E(H^{\mathbb{C}}/P_s))$ such that $\varphi \in H^0(E_\sigma(\mathfrak{m}_s) \otimes K)$, we have that* $\deg(E)(\sigma, s) - \langle \alpha, s \rangle > 0$.
- α-**polystable** *if it is α-semistable and for any $s \in i\mathfrak{h}_\iota$ and any holomorphic reduction $\sigma \in \Gamma(E(H^{\mathbb{C}}/P_s))$ and such that* $\deg(E)(\sigma, s) - \langle \alpha, s \rangle = 0$, *there is a holomorphic reduction of the structure group $\sigma_{L_s} \in \Gamma(E_\sigma(P_s/L_s))$ to a Levi subgroup L_s. Furthermore, under these hypothesis φ is required to belong to $H^0(E_\sigma(\mathfrak{m}_s^0) \otimes K) \subset H^0(E_\sigma(\mathfrak{m}_s) \otimes K)$.*

Remark 3.6. We may define a real group $G_{L_s} = (L_s \cap H)\exp(\mathfrak{m}_s^0 \cap \mathfrak{m})$ with maximal compact subgroup a compact real form $L_s \cap H$ of the complex group L_s and $\mathfrak{m}_s^0 \cap \mathfrak{m}$ as isotropy representation. Thus, an α- polystable G-Higgs bundle reduces to a G_{L_s}-Higgs bundle since φ belongs $H^0(E_\sigma(\mathfrak{m}_s^0) \otimes K)$. Moreover, this G_{L_s}-Higgs bundle is α-stable.

Remark 3.7. If we replace K in the definition of G-Higgs bundle by any holomorphic line bundle L on X, we get the notion of Higgs pair. More precisely, a L**-twisted** G**-Higgs pair** (E, φ) consists of a principal $H^{\mathbb{C}}$-bundle E, and a holomorphic section $\varphi \in H^0(E(\mathfrak{m}^{\mathbb{C}}) \otimes L)$. The notions of stability are as in Definition 3.5, replacing K by L.

Remark 3.8. Note that the notion of α-stability with $\alpha \neq 0$ only makes sense for groups of Hermitian type, since α belongs to the centre of $\mathfrak{h}$, which is not zero if and only if the centre of a maximal compact subgroup H is non-discrete, i.e., if G is of Hermitian type.

Remark 3.9. The differences of this stability condition with the one given in [GGM09] are very subtle. They concern the characters (and hence parabolic subgroups) to be considered in the α-stability, and the non-connectedness of the group G. From the point of view of geometric invariant theory, these issues have been treated in Section 2.7.5 and Remark 2.7.5.4 of [Sch08]. The definition of stability for a G-Higgs bundle given in [GGM09] for G connected involves the parabolic subgroups of $H^{\mathbb{C}}$, described as subgroups conjugated to some P_s. These subgroups P_s make sense for non-connected groups and the definition of stability in [GGM09] can be naturally extended. For non-connected groups, the subgroups P_s are the R-parabolics, as described in Section 2.5.1. On the other hand, in [GGM09], the conditions of stability are also stated in terms of filtrations of a vector bundle, thanks to an auxiliary faithful representation. This allows one to give simplified notions of stability for concrete cases, like $\mathrm{Sp}(2n,\mathbb{R})$, but it may not be possible for other groups, such as $\mathrm{Mp}(2n,\mathbb{R})$, which have no faithful finite dimensional representations (see Remark 2.8).

3.2 HK correspondence and moduli spaces

The notion of parameter-depending stability emerges from the study of the Hitchin equations. The equivalence between the existence of solutions to these equations and the α-polystability of Higgs bundles is known as the Hitchin-Kobayashi correspondence. A historical introduction, motivation and further study can be found in [DK90], [Kob87] and [LT95]. We use the formulation of [GGM09].

Theorem 3.10. *Let (E,φ) be a G-Higgs bundle over a Riemann surface X with volume form ω. Then (E,φ) is α-polystable if and only if there exists a reduction h of the structure group of E from $H^{\mathbb{C}}$ to H such that*

$$F_h - [\varphi, \tau_h(\varphi)] = \alpha\omega \tag{3.10.1}$$

where $\tau_h : \Omega^{1,0}(E(\mathfrak{m}^{\mathbb{C}})) \to \Omega^{0,1}(E(\mathfrak{m}^{\mathbb{C}}))$ is the combination of the anti-holomorphic involution in $E(\mathfrak{m}^{\mathbb{C}})$ defined by the compact real form at each point determined by h (see Section 2.1) and the conjugation of 1-forms, and F_h is the curvature of the unique H-connection compatible with the holomorphic structure of E (the Chern connection).

This theorem was proved by Hitchin in the case of $\mathrm{SL}(2,\mathbb{C})$, by Simpson for G complex, and is extended to a general reductive real group G as stated in [BGG06] and [GGM09].

The moduli space of α-polystable G-Higgs bundles $\mathcal{M}^\alpha(G)$ is by definition the set of isomorphism classes of α-polystable G-Higgs bundles (E,φ). We say that two G-Higgs bundles (E,φ) and (E',φ') are isomorphic if there is an isomorphism $f : E \to E'$ such that $\varphi' = f^*\varphi$, where f^* is the map $E(\mathfrak{m}^{\mathbb{C}}) \otimes K \to E'(\mathfrak{m}^{\mathbb{C}}) \otimes K$ induced by f. This space has the structure of a complex analytic variety, as can be seen by the standard slice method, which gives local models via the so-called Kuranishi map (see, e.g., [Kob87], [LT95]). When G is algebraic and under fairly general conditions, the moduli spaces $\mathcal{M}^\alpha(G)$ can be constructed by geometric invariant theory and hence are complex algebraic varieties. The recent work of Schmitt ([Sch05], [Sch08]) deals with the construction of the moduli space of L-twisted G-Higgs pairs for G a real reductive Lie group. This construction generalizes the constructions of the moduli space of G-Higgs bundles done by Ramanathan ([Ram75]) when G is compact, and by Simpson ([Sim94b], [Sim94a]) when G is a complex reductive algebraic.

Remark 3.11. The moduli space of α-polystable L-twisted G-Higgs pairs is denoted by $\mathcal{M}_L^\alpha(G)$.

3.3 G-Higgs bundles when G is of Hermitian type

Our goal in this thesis is to study G-Higgs bundles in the case in which G/H is a Hermitian symmetric space of the non-compact type. As seen in Chapter 2, this means that G/H admits a complex structure compatible with the Riemannian structure of G/H, making G/H a Kähler manifold. If G/H is irreducible, the centre of $\mathfrak{h}$ is one-dimensional and the almost complex structure on G/H is defined by a generating element in $J \in Z(\mathfrak{h})$ (acting through the isotropy representation on $\mathfrak{m}^{\mathbb{C}}$). This complex structure defines a decomposition $\mathfrak{m}^{\mathbb{C}} = \mathfrak{m}^+ + \mathfrak{m}^-$, where $\mathfrak{m}^+$ and $\mathfrak{m}^-$ are the i and the $-i$ eigenspaces of $\mathfrak{m}^{\mathbb{C}}$ respectively. Table C.1 in Appendix C shows these objects for the irreducible classical and exceptional Hermitian symmetric spaces.

Let now (E, φ) be a G-Higgs bundle over a compact Riemann surface X. The decomposition $\mathfrak{m}^{\mathbb{C}} = \mathfrak{m}^+ + \mathfrak{m}^-$ gives a vector bundle decomposition $E(\mathfrak{m}^{\mathbb{C}}) = E(\mathfrak{m}^+) \oplus E(\mathfrak{m}^-)$ and hence the Higgs field has two components.

$$\varphi = (\beta, \gamma) \in H^0(X, E(\mathfrak{m}^+) \otimes K) \oplus H^0(X, E(\mathfrak{m}^-) \otimes K) = H^0(X, E(\mathfrak{m}^{\mathbb{C}}) \otimes K).$$

Although the Higgs field is $\varphi = (\beta, \gamma)$, we will also refer to β and γ as Higgs fields.

Remark 3.12. When the group G has a complexification $G^{\mathbb{C}}$ (see Remark 2.9), it is useful to take a faithful representation of $G^{\mathbb{C}}$ to describe a G-Higgs bundle in terms of associated vector bundles. This is especially simple when G is a classical group and thus has the standard representation. Table C.2 describes in this fashion the G-Higgs bundles for the classical groups. This is the approach taken in [BGG06]. In Table C.3, we do the same for the exceptional Lie groups..

Example 3.13. *For $G = \mathrm{Sp}(2n, \mathbb{R})$, a maximal compact subgroup is given by $H = \mathrm{U}(n)$, and the isotropy representation is $\mathfrak{m}^{\mathbb{C}} = \mathrm{Sym}^2(\mathbb{C}^n) \oplus \mathrm{Sym}^2(\mathbb{C}^n)^*$, where $\mathbb{C}^n$ is the standard representation of $U(n)$. An $\mathrm{Sp}(2n, \mathbb{R})$-Higgs bundle (E, φ) consists then of a principal $\mathrm{GL}(n, \mathbb{C})$-bundle E together with the Higgs field φ. The bundle E is equivalent to a holomorphic rank n vector bundle V, and the field consists of* *two symmetric maps* $\beta : V \to V^* \otimes K$, $\gamma : V^* \to V \otimes K$. *Similarly, an* $\mathrm{SO}^*(2n)$*-Higgs bundle consists of a holomorphic rank* n *vector bundle* V*, together with skew-symmetric fields* $\beta : V \to V^* \otimes K$, $\gamma : V^* \to V \otimes K$.

Example 3.14. *Let* (E, φ) *be an* $\mathrm{SU}(p, q)$*-Higgs bundle. For* $G = \mathrm{SU}(p, q)$,

$$H^{\mathbb{C}} = S(\mathrm{GL}(p, \mathbb{C}) \times \mathrm{GL}(q, \mathbb{C})) := \{(A, B) \in \mathrm{GL}(p, \mathbb{C}) \times \mathrm{GL}(q, \mathbb{C}) \mid \det B = (\det A)^{-1}\},$$

and the principal $H^{\mathbb{C}}$*-bundle* E *is equivalent to two holomorphic vector bundles* V *and* W *or rank* p *and* q*, respectively, such that* $\det W = (\det V)^{-1}$. *The Higgs field consists of the two components* $\beta : W \to V \otimes K$ *and* $\gamma : V \to W \otimes K$.

Let (E, φ) be a G-Higgs bundle. Consider the Toledo character χ_T and the rational number q_T defined in Section 2.4.1 such that $q_T \cdot \chi_T$ lifts to a character $\tilde{\chi}_T$ of $H^{\mathbb{C}}$.

Definition 3.15. *We define the* ***Toledo invariant*** d *of* (E, φ) *by*

$$d = \frac{1}{q_T} \deg(E(\tilde{\chi}_T)).$$

Equivalently, given q *a positive integer multiple of* q_T *we have that*

$$d = \frac{1}{q} \deg(E(\tilde{\chi}_T^{q/q_T})).$$

We will study in Sections 3.5 and 5.2 the relation of this definition with other definitions given in the context of Higgs bundles as well as in the study of surface group representations.

We introduce now the notation E^K, which will be widely used from now on. Regard K as a principal $\mathbb{C}^*$-bundle, and consider the principal $H^{\mathbb{C}} \times \mathbb{C}^*$-bundle given by fibred product $E^K := E \times_X K$. The vector bundle $E(\mathfrak{m}^+) \otimes K$ is isomorphic to $E^K(\mathfrak{m}^+)$, where $H^{\mathbb{C}} \times \mathbb{C}^*$ acts on $\mathfrak{m}^+$ via $(h, \lambda) \cdot Y = \lambda \operatorname{Ad}(h) Y$ for $(h, \lambda) \in H^{\mathbb{C}} \times \mathbb{C}^*$ and $Y \in \mathfrak{m}^{\mathbb{C}}$. We define a notion of **rank** for the two components of the Higgs field as follows.

Definition 3.16. *Consider the field* $\beta \in H^0(E^K(\mathfrak{m}^+))$. *This field is equivalent to an* $\operatorname{Ad}(H^{\mathbb{C}} \times \mathbb{C}^*)$*-equivariant map* $f_\beta : E^K \to \mathfrak{m}^+$. *Take* $x \in X$. *By the equivariance and Proposition 2.49, for any* $p \in E^K$ *such that* $\pi(p) = x$*, the rank of* $f_\beta(p) \in \mathfrak{m}^+$ *only depends on* x *and we therefore have a well defined notion of rank of* β *at the point* x. *The* ***rank*** *of* β *is the same over almost all the points of* X*, we call this value rank of* β *and note it as* $\mathrm{rk}(\beta)$. *Analogously we define the rank of* $\gamma \in H^0(E^K(\mathfrak{m}^-))$, $\mathrm{rk}(\gamma)$.

The rank of a Higgs field may drop at a finite number of points of X. Let X_{reg} be the points in which the rank is constant, i.e., does not drop.

3.4 α-Milnor-Wood inequality

In this section we give a proof of a refined version of the Milnor-Wood inequality. The degree of the bundle is bounded by the ranks of the Higgs fields (which are in turn bounded by the rank of the symmetric space), the genus of the surface X and the parameter $\alpha \in i\mathfrak{z}$ given by the stability condition. Recall that $r := \mathrm{rk}(G/H)$.

Lemma 3.17. *Let the Higgs field β (equivalently γ) have rank $1 \leq r' \leq r$ and let $P_{r'}$ be a parabolic subgroup of $H^{\mathbb{C}}$ given by an element of rank r' in $\mathfrak{m}^+$ (as defined in Section 2.5.2). Then, the structure group of the principal $H^{\mathbb{C}}$-bundle E reduces to the parabolic subgroup $P_{r'}$.*

Proof. Consider the maps

$$\mathrm{ad}(\beta)_{|E(\mathfrak{h}^{\mathbb{C}})} : E(\mathfrak{h}^{\mathbb{C}}) \to E(\mathfrak{m}^+) \qquad \mathrm{ad}(\beta)_{|E(\mathfrak{m}^+)} : E(\mathfrak{m}^-) \to E(\mathfrak{h}^{\mathbb{C}})$$

and define the subset $F = \mathrm{Ker}(\mathrm{ad}(\beta)_{|E(\mathfrak{h}^{\mathbb{C}})}) \oplus \mathrm{Im}(\mathrm{ad}(\beta)_{|E(\mathfrak{m}^+)}) \otimes K^{-1} \subset E(\mathfrak{h}^{\mathbb{C}})$.

Take a trivialization of E over an open set $U \subset X$ given by a local section $\sigma : U \to E$. This section gives at the same time a trivialization for all the associated bundles to E, by the map $E(V)_{|U} \to U \times V$ defined by $[\sigma(x), v] \mapsto (x, v)$.

Let $\beta(x) = [\sigma(x), c(x)]$ over U. We have that F defines the subset

$$\bigcup_{x \in U} \{x\} \times \mathfrak{p}_{c(x)} \subset U \times \mathfrak{h}^{\mathbb{C}}.$$

Let $U_{reg} = U \cap X_{reg}$ be the regular points of U. The rank of $c(x)$ is the same over U_{reg}, and by Lemma 2.64, this defines a map $x \mapsto \mathfrak{p}_{c(x)}$ from U_{reg} to $Gr(\mathfrak{p}_{\mathrm{rk}(\beta)}, \mathfrak{h}^C)$, the Grassmanian of parabolic subalgebras of $\mathfrak{h}^{\mathbb{C}}$ isomorphic to one given by an element of rank $\mathrm{rk}(\beta')$. Also by Lemma 2.64, we know that $H^{\mathbb{C}}$ acts transitively in such parabolic subalgebras. Since the stabilizer of one of them is the parabolic subgroup $P_{r'}$, the Grassmanian is the homogeneous space $H^{\mathbb{C}}/P_{r'}$ and we have a map $U_{reg} \to H^{\mathbb{C}}/P_{r'}$.

We use now that $P_{r'}$ is a parabolic subgroup of $H^{\mathbb{C}}$ and therefore, the quotient $H^{\mathbb{C}}/P_{r'}$ is compact. This makes the limit of the images to points in $U \setminus U_{reg}$ lie again in $H^{\mathbb{C}}/P$ and we get a map $U \to H^{\mathbb{C}}/P_{r'}$.

Repeating this argument for a finite open cover of the compact Riemann surface X, we obtain a section of of the bundle $E(H^{\mathbb{C}}/P_{r'})$, i.e., a reduction of the structure group from $H^{\mathbb{C}}$ to $P_{r'}$. $\square$

The stability conditions defined in Section 3.1 involve a parameter $\alpha \in i\mathfrak{z}$. Since the element J giving the almost complex structure (Proposition 2.2) belongs to $\mathfrak{z}$

and, in the cases we are considering, $\mathfrak{z}$ is one dimensional, we have that $\alpha = i\lambda J$ with $\lambda \in \mathbb{R}$.

Theorem 3.18. *Let $\alpha \in i\mathfrak{z}$ such that $\alpha = i\lambda J$ for $\lambda \in \mathbb{R}$. Let (E, β, γ) be an α-semistable G-Higgs bundle. Then, the Toledo invariant $d = \frac{1}{q_T}\deg(E(\widetilde{\chi}_T))$ satisfies:*

$$-\operatorname{rk}(\beta)(2g-2) - \left(\frac{2\dim\mathfrak{m}}{N} - \operatorname{rk}(\beta)\right)\lambda \leq d \leq \operatorname{rk}(\gamma)(2g-2) + \left(\frac{2\dim\mathfrak{m}}{N} - \operatorname{rk}(\gamma)\right)\lambda,$$

where N is the dual Coxeter number and $\dim\mathfrak{m}$ is the dimension of the isotropy representation of G. In the tube-type case, this simplifies to:

$$-\operatorname{rk}(\beta)(2g-2) - (r - \operatorname{rk}(\beta))\lambda \leq d \leq \operatorname{rk}(\gamma)(2g-2) + (r - \operatorname{rk}(\gamma))\lambda.$$

Proof. Let $r' = \operatorname{rk}(\beta)$. For the sake of simplicity, let $P = P_{r'}$ and $\mathfrak{p} = \mathfrak{p}_{r'}$, following Remark 2.65. We consider the reduction σ of E from $H^{\mathbb{C}}$ to P given by Lemma 3.17 and call E_P the reduced bundle. From Section 2.5, we have that $\chi' = \chi_T - \chi_{\Gamma'}$ is a strictly antidominant character of $\mathfrak{p}$. Since we want the characters to lift to the group, consider q an integer multiple of $lcm(q_T, q_{\Gamma'})$, the least common multiple of q_T given by (2.39.1) and $q_{\Gamma'}$, given by Lemma 2.68. The character $q \cdot \chi'$ lifts to the character of P

$$\tilde{\chi} = (\tilde{\chi}_T)^{q/q_T}(\tilde{\chi}_{\Gamma'})^{-q/q_{\Gamma'}}.$$

We have that $\varphi \in H^0(E_P(\mathfrak{m}^-_\chi) \otimes K)$ is satisfied by Proposition 2.60. From the semistability condition we have that

$$\deg(E)(\sigma, \chi') - \langle \alpha, \chi' \rangle \ \geq 0. \tag{3.18.1}$$

As both $q\chi_T$ and $q\chi_{\Gamma'}$ are characters of the Lie algebra of P, we have that

$$q \cdot \deg(E)(\sigma, \chi') = \deg(E_P(\tilde{\chi}_T^{q/q_T})) - \deg(E_P(\tilde{\chi}_{\Gamma'}^{q/q_{\Gamma'}})),$$

where

$$\deg(E_P(\tilde{\chi}_T^{q/q_T})) = \deg(E(\tilde{\chi}_T^{q/q_T})) = q \cdot \frac{1}{q_T}\deg(E(\tilde{\chi}_T)) = q \cdot d,$$

where d is the Toledo invariant. Thus, from (3.18.1),

$$q \cdot d \geq \deg(E_P(\tilde{\chi}_{\Gamma'}^{q/q_{\Gamma'}})) + q \cdot \langle \alpha, \chi' \rangle. \tag{3.18.2}$$

On the other hand, we study $\deg(E_P(\tilde{\chi}_{\Gamma'}^{q/q_{\Gamma'}}))$. From the projections in diagram (2.68.1),

$$P_{\Gamma'} \to L_{\Gamma'} \to \widetilde{L_{\Gamma'}},$$

we extend the structure group of E_P to $\widetilde{L_{\Gamma'}}$. Let us call this new bundle $E_{\widetilde{L}}$. We have that $\tilde{\chi}_{\Gamma'}^{q/q_{\Gamma'}}$ is also a character of $\widetilde{L}$ and

$$\deg(E_P(\tilde{\chi}_{\Gamma'}^{q/q_{\Gamma'}})) = \deg(E_{\widetilde{L}}(\tilde{\chi}_{\Gamma'}^{q/q_{\Gamma'}})).$$

We regard β as an element in $H^0(E_{\widetilde{L}}(\mathfrak{m}_{\Gamma'}^+)\otimes K)$, where $\mathfrak{m}_{\Gamma'}^+$ is endowed with a Jordan algebra structure whose rank is r'. The determinant $\det_{\Gamma'}$ of $\mathfrak{m}_{\Gamma'}^+$ together with Lemma 2.69 gives a map

$$\det^q : E_{\widetilde{L}}(\mathfrak{m}_{\Gamma'}^-)\otimes K \to E_{\widetilde{L}}(\tilde{\chi}_{\Gamma'}^{q/q_{\Gamma'}})\otimes K^{q\cdot r'}.$$

Since the rank of β is r', we have that $\det^q(\beta)\in H^0(E_{\widetilde{L}}(\tilde{\chi}_{\Gamma'}^{q/q_{\Gamma'}})\otimes K^{qr'})$ does not vanish, and therefore defines a global section of $E_{\widetilde{L}}(\tilde{\chi}_{\Gamma'}^{q/q_{\Gamma'}})\otimes K^{qr'}$ which is non-zero everywhere. Hence, the degree of this bundle is positive, i.e.,

$$\deg(E_{\widetilde{L}}(\tilde{\chi}_{\Gamma'}^{q/q_{\Gamma'}})) \geq -qr'(2g-2). \tag{3.18.3}$$

Note that to deduce inequality (3.18.3) we have not used the hypothesis of the semistability of (E,β,γ).

From (3.18.2) and (3.18.3) we finally have, with the notation $\chi_\beta := \chi'$ and recalling that $r' = \mathrm{rk}(\beta)$,

$$d \geq -\mathrm{rk}(\beta)(2g-2) + \langle \alpha, \chi_\beta\rangle. \tag{3.18.4}$$

We compute $\langle \alpha, \chi_\beta\rangle$ for $\alpha = i\lambda J$ and $\chi_\beta = \chi_T - \chi_{\Gamma'}$:

$$\langle \alpha, \chi_\beta\rangle = (\chi_T - \chi_{\Gamma'})(i\lambda J) = \left(\frac{2}{N}\sum_{\alpha\in\Delta_Q^+}\alpha - \sum_{i=1}^{\mathrm{rk}(\beta)}\gamma_i\right)(i\lambda J) = -\left(\frac{2\dim\mathfrak{m}}{N} - \mathrm{rk}\,\beta\right)\lambda,$$

since $\alpha(J) = i$ for $\alpha\in\Delta_Q^+$ and $\gamma_i\in\Delta_Q^+$.

Analogously, for some character χ_γ related to the Higgs field γ, we obtain

$$d \leq \mathrm{rk}(\gamma)(2g-2) + \langle \alpha, \chi_\gamma\rangle \tag{3.18.5}$$

and we similarly compute

$$\langle \alpha, \chi_\gamma\rangle = \left(\frac{2\dim\mathfrak{m}}{N} - \mathrm{rk}\,\gamma\right)\lambda.$$

In the tube-type case we know that $\frac{2\dim\mathfrak{m}}{N} = r$, which gives the second inequality. □

Remark 3.19. In the non-tube case, as $\dim \mathfrak{m} = ar(r-1)/2 + r + br$ and $N = a(r-1) + b + 2$, we have that:

$$\frac{2\dim \mathfrak{m}}{N} = r(1 + \frac{b}{N}).$$

The rational number $\frac{b}{N}$ measures the difference between $\frac{2\dim \mathfrak{m}}{N}$ and the rank r. It equals $\frac{(q-p)p}{p+q}$ in the case of $\mathrm{SU}(p,q)$, $\frac{1}{2}$ in the case of $\mathrm{SO}^*(4r+2)$ and $\frac{2}{3}$ in the case of E_6^{-14}. Equivalently,

$$\frac{\frac{2\dim \mathfrak{m}}{N} - r}{r} = \frac{b}{N}.$$

In the tube-type case, using the determinant of the Higgs field we can prove the following proposition.

Proposition 3.20. *In the tube-type case, the Toledo invariant is maximal negative (resp. positive) if and only the Higgs field β (resp. γ) has maximal rank at every point.*

Proof. For the sake of simplicity, assume that the Toledo character exponentiates to the group $H^{\mathbb{C}}$. Let (E, β, γ) be a G-Higgs bundle with Toledo invariant d, and let $r = \mathrm{rk}(G/H)$ as usual. Consider the field $\det \beta \in H^0(E(\chi_T) \otimes K^r)$. This field has maximal degree at every point if only if the degree of the line bundle $E(\chi_T) \otimes K$ is zero, i.e., if $d + r(2g-2) = 0$, or equivalently, $d = -r(2g-2)$ is maximal negative. □

Theorem 3.18 gives a bound of the parameter λ depending on the Toledo invariant d of an α-semistable G-Higgs bundle, where $\alpha = i\lambda J$.

Proposition 3.21. *Let $\alpha = i\lambda J$. If (E, β, γ) is an α-semistable G-Higgs bundle with Toledo invariant d, then*

$$\max \left\{ \frac{d - \mathrm{rk}(\gamma)(2g-2)}{\frac{2\dim \mathfrak{m}}{N} - \mathrm{rk}(\gamma)}, \frac{-d - \mathrm{rk}(\beta)(2g-2)}{\frac{2\dim \mathfrak{m}}{N} - \mathrm{rk}(\beta)} \right\} \leq \lambda.$$

Remark 3.22. In the case of $\mathrm{SU}(p,q)$ ($p \neq q$), an upper bound for the parameter is found in [BGG03]. However, there is no upper bound in the cases of $\mathrm{SU}(p,p)$ and $\mathrm{Sp}(2n, \mathbb{R})$. Presumably, similar methods to those of $\mathrm{SU}(p,q)$ might be applied in our general situation to obtain an upper bound for λ in the non-tube type case. Another interesting phenomenon observed in the cases of $\mathrm{SU}(p,p)$ and $\mathrm{Sp}(2n, \mathbb{R})$ is that the condition of α-polystability stabilizes from a given α_M, i.e., α-polystability is equivalent to α_M-polystability for any $\alpha \geq \alpha_M$. Probably, this is also true for all tube-type groups.

3.5 The Toledo invariant and the topological class

For $H^{\mathbb{C}}$ connected, the topological classification of $H^{\mathbb{C}}$-bundles E on X is given by a characteristic class $c(E) \in \pi_1 H^{\mathbb{C}}$ as follows. From the exact sequence

$$1 \to \pi_1 H^{\mathbb{C}} \to \widetilde{H^{\mathbb{C}}} \to H^{\mathbb{C}} \to 1$$

we obtain the long exact sequence in cohomology and, in particular, the connection map

$$H^1(X, H^{\mathbb{C}}) \xrightarrow{c} H^2(X, \pi_1 H^{\mathbb{C}}),$$

where G and $\pi_1 G$ denote the sheaves of locally constant functions in G and $\pi_1 G$ respectively, the domain parameterizes equivalence classes of flat principal G-bundles on X, and the target is isomorphic to $\pi_1 G$ by the universal coefficient theorem since $\dim_{\mathbb{R}} X = 2$ and the fact that the fundamental group of a Lie group is Abelian. Moreover, $\pi_1 H^{\mathbb{C}} \cong \pi_1 H$ since H is a deformation retract of $H^{\mathbb{C}}$. This map thus associates a topological class in $\pi_1 H$ to any G-Higgs bundles on X. For a fixed $d \in \pi_1(H)$, the moduli space $\mathcal{M}_d^\alpha(G)$ is defined as the set of isomorphism classes of α-polystable G-Higgs bundles (E, φ) such that $c(E) = d$.

By the relation between the fundamental group and the centre of a Lie group, the topological class in $\pi_1 H$ will be of special interest when H has a non-discrete centre, i.e., when G is of Hermitian type. In this case, $\pi_1(H)$ is isomorphic to $\mathbb{Z}$ plus possibly a torsion group (among the classical groups, $\mathrm{SO}_0(2, n)$ is the only with torsion). In the introduction of [BGG03], the Toledo invariant for a bundle $E \in H^1(X, H^{\mathbb{C}})$ is defined as the projection of the topological class $d \in \pi_1(H)$ to the torsion-free part, $\mathbb{Z}$. On the other hand, the Toledo invariant was defined in Section 3.3 as the degree of the associated line bundle $E(\widetilde{\chi_T}) \in H^1(X, \mathbb{C}^*)$ via the Toledo character.

We compare the two invariants by the following diagram.

$$\begin{tikzcd}
\pi_1(H^{\mathbb{C}}) \arrow[r] & \widetilde{H^{\mathbb{C}}} \arrow[r, "\pi"] & H^{\mathbb{C}} \arrow[d, "\chi_T"] \\
\mathbb{Z} \arrow[r] & \mathbb{C} \arrow[r, "\exp"] & \mathbb{C}^*.
\end{tikzcd}$$

Define a homomorphism $\widetilde{H^{\mathbb{C}}} \to \mathbb{C}$ as follows. Regard an element $\gamma \in \widetilde{H^{\mathbb{C}}}$ as a loop $\gamma : \mathrm{U}(1) \to H^{\mathbb{C}}$. The image by χ_T is a loop $\chi_T(\gamma) : U(1) \to \mathbb{C}^*$ giving an element of $\mathbb{C}$, the universal cover of $\mathbb{C}^*$. By restriction, we obtain a homomorphism between the kernels, $\pi_1(H^{\mathbb{C}}) \to \mathbb{Z}$. This homomorphism would take the torsion part to $0 \in \mathbb{Z}$ and the generator of $\mathbb{Z} \cong \pi_1(H^C)$ to some integer $\mathbb{Z}$, $1 \in \mathbb{Z} \subset \pi_1(H^{\mathbb{C}}) \mapsto n \in \mathbb{Z}$.

From the completed diagram,

$$\begin{tikzcd}
\pi_1(H^{\mathbb{C}}) \arrow[r] \arrow[d, "f"] & \widetilde{H^{\mathbb{C}}} \arrow[r, "\pi"] \arrow[d, "\widetilde{\chi_T}"] & H^{\mathbb{C}} \arrow[d, "\chi_T"] \\
\mathbb{Z} \arrow[r] & \mathbb{C} \arrow[r, "\exp"] & \mathbb{C}^*,
\end{tikzcd}$$

we have the following commuting diagram in cohomology, which gives us the relation of the two definitions.

$$\begin{tikzcd}
H^1(X, H^{\mathbb{C}}) \arrow[r] \arrow[d] & H^2(X, \pi_1(H^{\mathbb{C}})) \cong \pi_1(H^{\mathbb{C}}) \arrow[d, "f"] \\
H^1(X, \mathbb{C}^*) \arrow[r] & H^2(X, \mathbb{Z}) \cong \mathbb{Z}.
\end{tikzcd}$$

The homomorphism $f : \pi_1(H^{\mathbb{C}}) \to \mathbb{Z}$ shows that the projection of the topological class to $\mathbb{Z}$ is an integer multiple of the Toledo invariant, which may be rational.

We compute this multiple by considering the image by χ_T of a loop generating $\pi_1 H^{\mathbb{C}}$. Recall that $H^{\mathbb{C}} = [H^{\mathbb{C}}, H^{\mathbb{C}}] \times_D Z_0^{\mathbb{C}}$, where $Z_0^{\mathbb{C}}$ is the identity component of the centre of $H^{\mathbb{C}}$ and $D = [H^{\mathbb{C}}, H^{\mathbb{C}}] \cap Z_0^{\mathbb{C}}$. Let $l = |D|$ and let J be the element of $\mathfrak{z}(\mathfrak{h})$ giving the almost complex structure on $\mathfrak{m}$. We have that $Z_0^{\mathbb{C}} \cong \{e^{2\pi\theta J}\}_{\theta\in\mathbb{R}}$. The curve $\sigma : [0,1] \to H^{\mathbb{C}}$, $\sigma : \theta \mapsto e^{\frac{2\pi\theta J}{l}}$ is a loop generating $\pi_1 H^{\mathbb{C}}$. Its image by χ_T is the loop $\chi_T \circ \sigma : [0,1] \to \mathbb{C}^*$, $\chi_T \circ \sigma : \theta \mapsto e^{2\pi \frac{2\dim\mathfrak{m}}{l\cdot N}\theta i}$, since $\chi_T(J) = \frac{2\dim\mathfrak{m}}{N}$. The factor relating the two definitions is then $\frac{2\dim\mathfrak{m}}{lN}$. One has then the following Milnor-Wood inequality as a consequence of Theorem 3.18.

Theorem 3.23. *Let $d' \in \mathbb{Z}$ the invariant of a G-Higgs bundle (E, β, φ) defined from the projection of the characteristic class $\pi_1 H^{\mathbb{C}} \to \mathbb{Z}$, as defined at the beginning of Section 3.5. Then,*

$$|d'| \leq \frac{lN}{2\dim\mathfrak{m}} \operatorname{rk}(G/H)(2g-2).$$

If one computes the constant $\frac{lN}{2\dim\mathfrak{m}}$ from Tables 2.1 and 2.2, Theorem 3.23 gives the Milnor Wood inequalities surveyed in [BGG06], which we reproduce in Table C.2.

Chapter 4

Maximal Toledo invariant

4.1 Milnor-Wood inequality

From now on, we will consider the parameter α in the definition of stability to be 0. For the sake of brevity, we will talk about **stability** of a G-Higgs bundle, meaning 0-stability, and analogously for **semistability** and **polystability**. The case $\alpha = 0$ is of special interest because the moduli space of polystable G-Higgs bundles over a Riemann surface X is homeomorphic to the moduli space of representations of $\pi_1 X$ into G, as we will show in Section 5.1.

When $\alpha = 0$, we have that $\lambda = 0$ in Theorem 3.18, and obtain the inequality

$$-\operatorname{rk}(\beta)(2g-2) \leq d \leq \operatorname{rk}(\gamma)(2g-2).$$

Moreover, since both $\operatorname{rk}(\gamma)$ and $\operatorname{rk}(\beta)$ are bounded by $\operatorname{rk}(G/H)$, we obtain the usual Milnor-Wood inequality as a consequence.

Theorem 4.1 (Milnor-Wood inequality)**.** *Let G be a simple group of Hermitian type. Let d be the Toledo invariant of a semistable G-Higgs bundle. Then,*

$$|d| \leq \operatorname{rk}(G/H)(2g-2).$$

We define a Higgs bundle (E, φ) to be **maximal** if its Toledo invariant d attains one of the bounds of the Milnor-Wood inequality, i.e., $d = \pm \operatorname{rk}(G/H)(2g-2)$.

We focus on the maximal case because of the interesting phenomena that appear: rigidity and new invariants. From the point of view of representations, we have that maximal representations are discrete and faithful, and furthermore, Anosov. The space of maximal G-Higgs bundles shares many of the features of the Teichmüller space, which is indeed obtained as the maximal component for the group $\mathrm{SL}(2, R)$, so it can be regarded as a generalization of the Teichmüller space, as Hitchin pointed out in [Hit92].

For groups of tube type we prove the Cayley correspondence in Section 4.2. For groups of non-tube type, we prove in Section 4.3 that all objects are strictly polystable and reduce to the normalizer of a maximal tube. The rigidity theorem for groups of non-tube type will be formulated in terms of surface group representations in Section 5.3, where we will also review some of the interesting features of maximal Toledo representations.

As a convention, we use the notation $d_{max} = -\operatorname{rk}(G/H)(2g/2)$.

Proposition 4.2. *For groups G of tube-type, the moduli space $\mathcal{M}_{d_{max}}(G)$ is homeomorphic to the moduli space $\mathcal{M}_{-d_{max}}(G)$.*

Proof. Consider the root decomposition of $\mathfrak{g}^{\mathbb{C}}$,

$$\mathfrak{g}^{\mathbb{C}} = \mathfrak{t}^{\mathbb{C}} \oplus \bigoplus_{\alpha\in\Delta^+} \mathfrak{g}^{\mathbb{C}}_{\alpha} \oplus \bigoplus_{\alpha\in\Delta^-} \mathfrak{g}^{\mathbb{C}}_{\alpha},$$

where $\Delta = \Delta(\mathfrak{g}^{\mathbb{C}}, \mathfrak{t}^{\mathbb{C}})$. Take elements $e_\alpha, e_{-\alpha}, h_\alpha$ as in Section 2.1. We define an involution ψ in $\mathfrak{g}^{\mathbb{C}}$ by $\psi(e_\alpha) = e_{-\alpha}$, and h_α to $h_{-\alpha}$. This is consistent with the commutation relations and thus defines a Lie algebra isomorphism of $\mathfrak{g}^{\mathbb{C}}$. This isomorphism ψ leaves $\mathfrak{h}^{\mathbb{C}}$ invariant and sends $\mathfrak{m}^+$ to $\mathfrak{m}^-$ (and viceversa). The restriction of ψ to $\mathfrak{h}^{\mathbb{C}}$ exponentiates to an isomorphism of $H^{\mathbb{C}}$ which we denote also by ψ. The action of ψ on $\mathcal{M}_{d_{max}}(G)$ defines a new G-Higgs bundle $(\psi(E), \psi(\gamma), \psi(\beta))$.

From the maximality of (E, β, γ) we have that β has maximal rank at every point. Hence, $\psi(\beta)$ has maximal rank at every point. By Proposition 3.20, the Toledo invariant of $(\psi(E), \psi(\gamma), \psi(\beta))$ is maximal positive. This process is clearly invertible and thus defines an isomorphism between $\mathcal{M}_{d_{max}}(G)$ and $\mathcal{M}_{-d_{max}}(G)$. □

Example 4.3. *We describe this map for* $\mathrm{Sp}(2n, \mathbb{R})$*-Higgs bundles using the realization as vector bundles shown in Example 3.13. To the maximal triple (V, β, γ), with $\beta : V \to V^* \otimes K$ an isomorphism, it corresponds the maximal triple (V^*, γ^t, β^t) with $\beta^t : V^* \to V \otimes K$ an isomorphism.*

Remark 4.4. We believe the result to be true also in the non-tube type case, but we cannot provide a similar proof since Proposition 3.20 uses the determinant on the isotropy representation, which is not defined in the non-tube-type case. The case of $\mathrm{SU}(p, q)$ can be checked by using the vector space realization in Example 4.3, (V, W, β, γ) is sent to $(V^*, W^*, \gamma^t, \beta^t)$. Equivalently for $\mathrm{SO}^*(4n + 2)$, the correspondence is given by sending (V, β, γ) to (V^*, γ^t, β^t). A similar argument is used for E_6^{-14} by using the vector bundle realization given in Table C.3.

Remark 4.5. In the classical cases, both tube and non-tube type, the isomorphism is proved for a not necessarily maximal value of the Toledo invariant. One has that $\mathcal{M}_d(G) \cong \mathcal{M}_{-d}(G)$ for all d. Presumably, this is true in general.

Remark 4.6. Inner automorphisms leave invariant the components of the moduli space. Hence, the existence of an isomorphism sending maximal positive Toledo invariant to maximal negative Toledo invariant implies the existence of an outer automorphism of the group G. In fact, his can be checked case by case using the classification theorem: all the Dynkin diagrams of the groups of Hermitian type possess such a symmetry.

Since the results we are going to prove will be valid both for $\mathcal{M}_{d_{max}}(G)$ and $\mathcal{M}_{-d_{max}}(G)$ we use the notation $\mathcal{M}_{max}(G)$.

4.2 Tube-type groups and Cayley correspondence

In this section we imbed the moduli space of polystable G-Higgs bundles with maximal Toledo invariant $\mathcal{M}_{\max}(G)$ as a subvariety of the moduli space of polystable K^2-twisted H^*-Higgs pairs $\mathcal{M}_{K^2}(H^*)$.

Theorem 4.7. *Let G be a simple Hermitian group of tube type and H be a maximal compact subgroup. Let H^* be the non-compact dual of H in $H^{\mathbb{C}}$. Let J be the element in the centre of the Lie algebra $\mathfrak{g}$ giving the almost complex structure on $\mathfrak{m}$ (see Proposition 2.2). If the order of $e^{2\pi J} \in H^{\mathbb{C}}$ divides $(2g-2)$, then there is an injection of complex algebraic varieties*

$$\mathcal{M}_{\max}(G) \to \mathcal{M}_{K^2}(H^*). \tag{4.7.1}$$

Moreover, stable G-Higgs bundles correspond to stable K^2-twisted H^*-Higgs pairs.

For the classical groups, the injection of Theorem 4.7 is indeed an isomorphism as surveyed by Bradlow, García-Prada and Gothen in [BGG06]. We strongly believe that this is true in general. In Remark 4.12 we will explain a possible strategy to complete the proof.

The isomorphism was referred to as the *Cayley correspondence* inspired by the fact that the space is realized as a tube domain via the Cayley transform described in Section 2.2.

Remark 4.8. Tables 2.1 and 2.3 show that the hypothesis $o(e^{2\pi J})|(2g-2)$ is satisfied in the classical and exceptional cases. There may be problems in coverings of these groups, where $o(e^{2\pi J})$ may be bigger.

This result is interpreted as a rigidity result for Higgs bundles since the structure group of the K^2 object H^* is smaller and reveals new invariants coming from the group H^*. For example, when $G = \mathrm{Sp}(2n, \mathbb{R})$, we have that $H^* = \mathrm{GL}(n, \mathbb{R})$ with $H' = \mathrm{O}(n)$ as a maximal compact subgroup. To a $\mathrm{Sp}(2n, \mathbb{R})$-Higgs bundle we can thus attach the first and second Stiefel-Whitney classes of the principal $O(n, \mathbb{C})$-bundle given by the corresponding $GL(n, \mathbb{R})$-Higgs pair via the Cayley correspondence. In general, similar invariants may come from the non-connectedness and non-simply connectedness of H^*. In fact, following Tables C.2 and C.3, we see that the former always occurs for classical and exceptional groups.

Here we present a proof not depending on the classification theorem which outlines the role played by the Jordan algebra structure of $\mathfrak{m}^{\pm}$. This applies to all the groups including the four classical families, the two exceptional cases, as well as all their finite coverings and quotients.

From now on, let G a simple group of Hermitian type, and let H be a maximal compact subgroup.

Proposition 4.9. *There is a bijective correspondence between G-Higgs bundles (E, β, γ) such that* $\mathrm{rk}(\beta)$ *(resp.* $\mathrm{rk}(\gamma)$*) is maximal at every point and K^2-twisted H^*-Higgs pairs.*

Proof. We will construct a principal $H'^{\mathbb{C}}$-bundle F out of the principal bundle E and the Higgs field β. To do this, regard K as a principal $\mathbb{C}^*$-bundle, and consider the principal $H^{\mathbb{C}} \times \mathbb{C}^*$-bundle $E^K = E \times_X K$ as mentioned before Definition 3.16. By Lemma 2.5 we have that $\lambda \in \mathbb{C}^* \cong_\mu Z(H^{\mathbb{C}}) \subset H^{\mathbb{C}}$ acts on $\mathfrak{m}^+$ by multiplication. Use this to define an action of $\mathbb{C}^*$ on E^K by

$$\lambda \cdot (e, k) = (e \cdot \mu(\lambda), \lambda^{-s} k), \tag{4.9.1}$$

where $s = o(e^{2\pi J})$. We consider the quotient manifold $E^K/\mathbb{C}^*$, consisting of equivalence classes $[(e, k)]$ with $e \in E$, $k \in K$. This manifold becomes a principal $H^{\mathbb{C}}$-bundle when we can take a root of order s of the line bundle K, which is possible if and only if s divides $\deg L = 2g - 2$, as the hypothesis assures. Indeed, if the transition functions of E and K are given by cocycles $\{g_{\alpha,\beta}\}$ and $\{c_{\alpha\beta}\}$, the transition functions of $E^K/\mathbb{C}^*$ are given by $\{g_{\alpha\beta} c^s_{\alpha\beta}\}$. Moreover, the action $[(e, k)] \cdot h = [(eh, k)]$ is free, and $(E^K/\mathbb{C}^*)/H^{\mathbb{C}}$ is X, so the manifold $E^K/\mathbb{C}^*$ becomes a principal $H^{\mathbb{C}}$-bundle over X.

The associated vector bundle $E(\mathfrak{m}^+)$ is given by

$$E(\mathfrak{m}^{\mathbb{C}}) = E \times_{H^{\mathbb{C}}} \mathfrak{m}^{\mathbb{C}} = \frac{\{(e, m) \mid e \in E, m \in \mathfrak{m}^{\mathbb{C}}\}}{(e, m) \sim (eh^{-1}, \mathrm{Ad}(h)m)},$$

and we write its elements as equivalence classes $[e, m]$. Recall that the vector bundle $E(\mathfrak{m}^+) \otimes K$ is isomorphic to $E^K(\mathfrak{m}^+)$, where $H^{\mathbb{C}} \times \mathbb{C}^*$ acts on $\mathfrak{m}^+$ via $(h, \lambda) \cdot X = \lambda \operatorname{Ad}(h)X$ for $(h, \lambda) \in H^{\mathbb{C}} \times \mathbb{C}^*$ and $X \in \mathfrak{m}^{\mathbb{C}}$. By the definition of rank, since $\operatorname{rk}(\beta)$ is maximal and $H^{\mathbb{C}}$ acts transitively on the subset of $\mathfrak{m}^+$ consisting of non-zero determinant elements $\mathfrak{m}^+_{D\neq 0}$ (Proposition 2.49), we have that β is a holomorphic section of $E^K(\mathfrak{m}^+_{D\neq 0})$. Note that $E^K(\mathfrak{m}^+_{D\neq 0})$ is no longer a vector bundle . This section is given by an $H^{\mathbb{C}} \times \mathbb{C}^*$-equivariant map $f_\beta : E^K \to \mathfrak{m}^+_{D\neq 0} \subset \mathfrak{m}^+$ satisfying $f_\beta(e \cdot \mu(\lambda), \lambda^{-1}k) = f_\beta(e, k)$, for $\lambda \in \mathbb{C}^*$. This implies that f_β is indeed defined on the quotient $E^K/\mathbb{C}^*$, and is $H^{\mathbb{C}}$-equivariant, so we obtain

$$\bar{\beta} : E^K/\mathbb{C}^* \to \mathfrak{m}^+_{D\neq 0} \subset \mathfrak{m}^+.$$

From Lemma 2.51, $\mathfrak{m}^+_{D\neq 0} \cong H^{\mathbb{C}}/H'^{C}$, and we thus have a reduction of the structure group of $E^K/\mathbb{C}^*$ from $H^{\mathbb{C}}$ to H'^{C}, which yields a principal $H'^{\mathbb{C}}$-bundle F.

We next define a K^2-valued Higgs field for this bundle. Let $\mathfrak{h}^* = \mathfrak{h}' + i\mathfrak{m}'$ be the Cartan decomposition of the non-compact dual of $\mathfrak{h}$. We want to define $\gamma' \in H^0(F(\mathfrak{m}'^{\mathbb{C}}) \otimes K^2)$. Let us go back to our G-Higgs bundle with maximal Higgs field β. Recall that $\mathfrak{m}^+$ is endowed with a Jordan algebra structure and therefore has a determinant polynomial, det, of degree equal to the rank r of the symmetric space. The semi-invariance of det by the adjoint action of $H^{\mathbb{C}}$ on $\mathfrak{m}^+$ is described by the Toledo character χ_T, $\det(h \cdot X) = \chi_T(h) \det(X)$, for $X \in \mathfrak{m}^+$ and $h \in H^{\mathbb{C}}$ (see Section 2.4).

The properties of the determinant and its polarization allow us to define

$$\det : E(\mathfrak{m}^+) \to E_{\chi_T} = L, \qquad C : E(\mathfrak{m}^+)^{\otimes r} \to E_{\chi_T} = L,$$

for which we consider K-twisted versions:

$$\det : E(\mathfrak{m}^+) \otimes K \to L \otimes K^r \qquad C : (E(\mathfrak{m}^+) \otimes K)^{\otimes r} \to L \otimes K^r.$$

By hypothesis, $\beta \in H^0(E(\mathfrak{m}^+) \otimes K)$ is regular for every point, i.e., $\det(\beta) \neq 0 \in L \otimes K^r$ and thus defines a section which vanishes nowhere. Hence $L \otimes K^r \cong \mathcal{O}$, and we define a pairing

$$C(\underbrace{\beta, \ldots, \beta}_{r-2}, \_, \_) : (E(\mathfrak{m}^+) \otimes K) \otimes (E(\mathfrak{m}^+) \otimes K) \to \mathcal{O}.$$

Moreover, since $\mathfrak{m}^+$ and $\mathfrak{m}^-$ are dual to each other, $E(\mathfrak{m}^+)^* \cong E(\mathfrak{m}^-)$, and we have that $E(\mathfrak{m}^-) \otimes K^{-1} \cong E(\mathfrak{m}^+) \otimes K$. Tensoring with K^2 we get that

$$E(\mathfrak{m}^-) \otimes K \cong (E(\mathfrak{m}^+) \otimes K) \otimes K^2. \tag{4.9.2}$$

We describe now an isomorphism $F(\mathfrak{m}'^{\mathbb{C}}) \cong E^K(\mathfrak{m}^+)$ based on the $\mathrm{Ad}(H_0'^{\mathbb{C}})$-equivariant isomorphism $\psi : \mathfrak{m}'^{\mathbb{C}} \cong \mathfrak{m}^+$ given by $\frac{1}{2}\operatorname{ad} e_\Gamma$ in Lemma 2.18. Take an open subset $U_\alpha \subset X$ with trivializations $\sigma : U_\alpha \to E$, $\tau : U_\alpha \to K$ of E and K, such that $[\sigma(x), \tau(x)] \in F$. We have then associated trivializations of $F(\mathfrak{m}'^{\mathbb{C}})$ and $E^K(\mathfrak{m}^+)$: $[[\sigma(x), \tau(x)], m] \mapsto (x, m) \in U_\alpha \times \mathfrak{m}'^{\mathbb{C}}$ and $[(\sigma(x), \tau(x)), m] \mapsto (x, m) \in U_\alpha \times \mathfrak{m}^+$. Note that in the first case the brackets refer to $H'^{\mathbb{C}}$ equivalence classes, while in the second refer to $H^{\mathbb{C}} \times \mathbb{C}^*$ equivalence classes. We use the trivializations to lift the trivial isomorphism $\mathrm{Id} \times \psi : U_\alpha \times \mathfrak{m}'^{\mathbb{C}} \to U_\alpha \times \mathfrak{m}^+$ and define the isomorphism in the open set U_α by

$$[[\sigma(x), \tau(x)], m] \mapsto [(\sigma(x), \tau(x)), \psi(m)].$$

This map is well defined because of the $\mathrm{Ad}(H'^{\mathbb{C}})$-equivariance of ψ. If we take two open sets U_α, U_β, this definition is compatible in the intersection, so we define the isomorphism $F(\mathfrak{m}'^{\mathbb{C}}) \cong E^K(\mathfrak{m}^+)$ on X by taking an open cover.

Finally, from $\gamma \in H^0(E(\mathfrak{m}^-) \otimes K)$, using the isomorphism (4.9.2) and $F(\mathfrak{m}'^{\mathbb{C}}) \cong E^K(\mathfrak{m}^+)$, we get $\gamma' \in H^0(F(\mathfrak{m}'^{\mathbb{C}}) \otimes K^2)$.

To prove the converse, let (F, γ') be an H^*-Higgs bundle consisting of a principal $H'^{\mathbb{C}}$-bundle together with the Higgs field $\gamma' \in H^0(F(\mathfrak{m}'^{\mathbb{C}}) \otimes K^2)$. Consider $\overline{F} = F \times_l H^{\mathbb{C}}$, the bundle obtained by extension of the structure group to $H^{\mathbb{C}}$, on which $H'^{\mathbb{C}}$ acts by left multiplication l. The reduction from $\overline{F}$ to F is given by the map

$$\sigma : F \times_l H^{\mathbb{C}} \to H^{\mathbb{C}}/H'^{\mathbb{C}}$$
$$[(f, h)] \mapsto hH'^{\mathbb{C}}.$$

First, we obtain a principal $H^{\mathbb{C}}$-bundle by considering the manifold

$$E := (\overline{F} \times_X K^{-1})/\mathbb{C}^*,$$

where $\mathbb{C}^*$ acts analogously to formula (4.9.1). Moreover, it satisfies $(E \times_X K)/\mathbb{C}^* = \overline{F}$, as can be easily checked using transition functions. Thus, it is the inverse of the transformation defined above.

Second, we see the reduction as a map $\sigma : (E \times_X K)/\mathbb{C}^* \to H^{\mathbb{C}}/H'^{\mathbb{C}}$, from which we get a Higgs field β as the composition with the projection $E \times_X K \to E \times_X K/\mathbb{C}^*$ and the isomorphism $H^{\mathbb{C}}/H'^{\mathbb{C}} \cong \mathfrak{m}^+_{D\neq 0}$ (Lemma 2.51):

$$\beta : E \times_X K \to \mathfrak{m}^+_{D\neq 0} \subset \mathfrak{m}^+.$$

We define the Higgs field γ as the image of $\gamma' \in H^0(F(\mathfrak{m}'^{\mathbb{C}}) \otimes K^2)$ by the isomorphism $F(\mathfrak{m}'^{\mathbb{C}}) \otimes K^2 \cong E(\mathfrak{m}^+) \otimes K$ shown above. This operation on the Higgs fields is clearly the inverse of the defined above. $\square$

We study how the transformations performed in the previous lemma affect the stability.

Proposition 4.10. *Given a maximal (poly,semi)-stable G-Higgs bundle (E,φ), the corresponding K^2-twisted H^*-Higgs pair $(F,\gamma' \in H^0(F(\mathfrak{m}'^{\mathbb{C}}) \otimes K^2))$ given by Proposition 4.9 is (poly,semi)-stable.*

Proof. In the correspondence between the two objects, there is an intermediate object, the $\mathcal{O}$-twisted G-Higgs pair $(E^K/\mathbb{C}^*, \varphi \in H^0(E^K/\mathbb{C}^*(\mathfrak{m}^{\mathbb{C}})))$. However, the (poly,semi)-stability of this object has a parameter not equal to zero, but shifted by the quotient of the degree of the canonical bundle by the constant $l = |[H^{\mathbb{C}}, H^{\mathbb{C}}] \cap Z_0^{\mathbb{C}}$.

Antidominant characters of parabolic subgroups of $H'^{\mathbb{C}}$ are given by elements $s \in i\mathfrak{h}'$. By the inclusion in $i\mathfrak{h}$, s determines a parabolic subgroup P_s of $H^{\mathbb{C}}$ and an antidominant character of P_s.

We check with some detail that a reduction of F comes from a reduction of E. A reduction of F from $H'^{\mathbb{C}}$ to P' can be extended H-equivariantly using the injection $H'^{\mathbb{C}}/P' \to H^{\mathbb{C}}/P$, and this yields a reduction of $E^K/\mathbb{C}^*$ from $H^{\mathbb{C}}$ to P. We now give an $\mathrm{Ad}(H^{\mathbb{C}})$-equivariant isomorphism of E and $E^K/\mathbb{C}^*$. Given an open set U_α where E trivializes as $\sigma : U_\alpha \to H^{\mathbb{C}}$ and K as $\tau : U_\alpha \to \mathbb{C}^*$, we have that E^K trivializes as $\sigma \times \tau : U_\alpha \to H^{\mathbb{C}} \times \mathbb{C}^*$. We obtain a trivialization of $E^K/\mathbb{C}^*$ by the map $H^{\mathbb{C}} \times \mathbb{C}^* \to H^{\mathbb{C}}$ defined by $(h,\lambda) \mapsto h\mu(\lambda^{-1})$, where μ was defined in Lemma 2.5. By this isomorphism, we see that the reductions of E are the same as the reductions of $E^K/\mathbb{C}^*$. Recall that a reduction of a G-bundle F to $G' \subset G$ is given by a section of $F(G/G')$, or equivalently a G-equivariant map $F \to G/G'$. Both reductions of E and $E^K/\mathbb{C}^*$ are given locally by maps $U_\alpha \times H^{\mathbb{C}} \to H^{\mathbb{C}}/P$ or $H^{\mathbb{C}}/L$ for some parabolic or Levi subgroup. In the same way we have a correspondence of the condition on the Higgs field: if $\varphi \in H^0(E^K(\mathfrak{m}_s))$ is given locally by $H^{\mathbb{C}} \times \mathbb{C}^*$-equivariant maps $U_\alpha \times H^{\mathbb{C}} \times \mathbb{C}^* \to \mathfrak{m}_s$, we get $H^{\mathbb{C}}$-equivariant maps $U_\alpha \times H^{\mathbb{C}} \to \mathfrak{m}_s$ giving $\varphi \in H^0(E^K/\mathbb{C}^*(\mathfrak{m}_s))$. The converse is proved in the same way, using the inclusion $H^{\mathbb{C}} \to H^{\mathbb{C}} \times \mathbb{C}^*$.

The numerical conditions on parabolic subgroups and antidominant characters of $H'^{\mathbb{C}}$ for F are then translated to conditions on parabolic subgroups and antidominant characters of $H^{\mathbb{C}}$ for E, and the semistability of E gives the semistability of F.

For stability and polystability, we must consider $\mathfrak{h}_\iota$ and $\mathfrak{h}'_{\iota'}$, where $d\iota' : \mathfrak{h}' \to \mathrm{End}(\mathfrak{m}'^{\mathbb{C}})$ is the complexified isotropy representation of $\mathfrak{h}^*$. In this case, we would get an extra numerical condition to check for $(E^K/\mathbb{C}^*, \varphi)$ if and only if there was some $s \in i\mathfrak{h}' = i\mathfrak{h}_\iota \cap i\mathfrak{h}'$ such that $s \notin i\mathfrak{h}'_\iota$. But this would imply $s \in i\mathfrak{z}(\mathfrak{h}') \cap \ker(d\iota)$, so we would have $s \in \mathfrak{z}(\mathfrak{h})$. But the centre of $\mathfrak{h}$ is not inside $\mathfrak{h}'$ because it does not annihilate ie_Γ. In fact, the proof of Theorem 4.1.11 in [KW65] gives us that in the tube type case $i\mathfrak{h}' \subset i[\mathfrak{h}, \mathfrak{h}]$. This equivalence between the elements s giving antidominant characters, together with the correspondence of reductions to Levi subgroups gives the equivalence between stability and polystability. □

Remark 4.11. Although a general approach has been used in the proof of the two preceding propositions, more can be said about the subalgebras $\mathfrak{h}_\iota$ and $\mathfrak{h}'_\iota$. By Remark 3.4 we have that $\mathfrak{z}' = 0$ for $\mathfrak{h}$, so $\mathfrak{h}_\iota = \mathfrak{h}$. For $\mathfrak{h}'$, it can be checked case by case that $\mathfrak{h}'$ is centreless, so again $\mathfrak{h}'_\iota = \mathfrak{h}'$.

Remark 4.12. In order to have an isomorphism, we need a converse of Proposition 4.10. It would remain to prove that any (poly,semi)-stable K^2-twisted H^*-Higgs pairs comes from a (poly,semi)-stable G-Higgs bundle. Lemma 2.66 gives a correspondence between characters χ'_s of parabolic subgroups P'_s of $H'^{\mathbb{C}} = Stab_{H^{\mathbb{C}}}(m)$ and characters χ_s of parabolic subgroups P_s of $H^{\mathbb{C}}$ such that $m \in \mathfrak{m}^0_s$. Since the Higgs field sits in $\mathfrak{m}_s$ pointwise, and not in $\mathfrak{m}^0_s$, we cannot use only this to prove that the stability of F implies the stability of E. Although in Proposition 2.60 is proved that $\mathfrak{m}^+ \cap \mathfrak{m}_s = \mathfrak{m}^+ \cap \mathfrak{m}^0_s$, this only works for the antidominant character χ_s defined in Section 2.5.2, and not for any arbitrary antidominant character.
A possible strategy to prove this fact would be to show that there exist a maximal destabilizing parabolic subgroup and reduction for any non polystable maximal $H^{\mathbb{C}}$-bundle and that these induce a maximal destabilizing parabolic subgroup and reduction for the corresponding K^2-twisted H^*-Higgs pair. Hence, there would be no (poly)-stable K^2-twisted H^*-Higgs pairs coming from non (poly)-stable G-Higgs bundleand we will have the isomorphism. Details of this strategy has been worked out for the case of $\mathrm{Sp}(2n, \mathbb{R})$ in terms of filtrations and the simplified notion of stability ([GGM12]). In this case, from a maximal destabilizing parabolic subgroup and reduction for the polystable $\mathrm{GL}(n, \mathbb{C})$-bundle, one defines two parabolic subgroups and reductions of the corresponding Higgs pair, in such a way that one of them violates the stability condition.

We have now the two main ingredients to provide a proof of the Cayley correspondence.

Proof of Theorem 4.7. Given (E, β, γ) a polystable (resp. stable) G-Higgs bundle. If the Toledo invariant is maximal negative, then we have that β is regular in every point. We apply Proposition 4.9 to get a K^2-twisted $H'^{\mathbb{C}}$-Higgs pair which is polystable (resp. stable) by Proposition 4.10. Since this correspondence is preserved by equivalence of bundles and pairs, we have an injection between the moduli spaces. By the existence of local universal families (see [Sch08]) the injection is indeed an imbedding of complex algebraic varieties. □

In the preexistent casewise proofs ([BGG06]), many of the geometrical ingredients were identified but not explicitly used. Our new proof shows how the Jordan algebra structure of the isotropy representation and the geometry of the domain interact to yield the Cayley correspondence. Moreover, this result generalizes the work of [BGG06] for classical groups in two ways. First, by considering quotients and coverings of the classical groups, even though they may not be matrix groups, as it is pointed out in Remark 2.9. And second, by including the exceptional case, stated as follows.

Theorem 4.13. *There exists an imbedding of complex algebraic varieties*

$$\mathcal{M}_{max}(\mathrm{E}_7^{-25}) \to \mathcal{M}_{K^2}(\mathrm{E}_6^{-26} \ltimes \mathbb{R}^*)$$

A maximal compact subgroup of $H^* = E_6^{-26} \ltimes \mathbb{R}^*$ is given by $H' = \mathrm{F}_4 \times \mathbb{Z}_2$. Since $H'^{\mathbb{C}}$ is non-connected, we consider the short exact sequence $1 \to H_0'^{\mathbb{C}} \to H'^{\mathbb{C}} \to \pi_0(H'^{\mathbb{C}}) \cong \mathbb{Z}_2 \to 1$ and the following homomorphism of its induced long exact sequence in cohomology,

$$H^1(X, H'^{\mathbb{C}}) \to H^1(X, \pi_0(H'^{\mathbb{C}})) \cong \mathbb{Z}_2^{2g}.$$

This map associates an invariant in $\mathbb{Z}_2^{2g}$ to any K^2-twisted H^*-Higgs pair, and hence to any G-Higgs bundle. This implies that $\mathcal{M}_{K^2}(\mathrm{E}_6^{-26} \ltimes \mathbb{R}^*)$ has at least 2^{2g} connected components. If the image of $\mathcal{M}_{max}(\mathrm{E}_7^{-25})$ inside $\mathcal{M}_{K^2}(\mathrm{E}_6^{-26} \ltimes \mathbb{R}^*)$ meets different connected components, or more strongly, the two moduli spaces are isomorphic, as we believe, we can give a bound for the number of connected components of $\mathcal{M}_{max}(\mathrm{E}_7^{-25})$.

The Cayley correspondence can be adapted to L-twisted Higgs bundles as follows. First, a version of the inequality of Milnor-Wood for an L-twisted Higgs bundle gives an invariant d_L bounded by $|d_L| \leq \mathrm{rk}(G/H) \deg L$. Let $\mathcal{M}_{L,\mathrm{max}}(G)$ be the moduli space of L-twisted G-Higgs bundles with maximal invariant d_L, and $\mathcal{M}_{L^2}(H^*)$ the moduli space of L^2-twisted H^*-Higgs bundles. In particular, if the map of Theorem 4.13 is an isomorphism, we have that $\mathcal{M}_{max}(\mathrm{E}_7^{-25})$ has at least 2^{2g} connected components.

Theorem 4.14. *Let G be a simple Hermitian group of tube type and H be a maximal compact subgroup. Let H^* be the non-compact dual of H in $H^{\mathbb{C}}$. Let J be the element* *in the centre of the Lie algebra $\mathfrak{g}$ giving the almost complex structure on $\mathfrak{m}$ (see 2.2). If the order of $e^{2\pi J} \in H^{\mathbb{C}}$ divides $\deg L$, then there is an imbedding of complex algebraic varieties*

$$\mathcal{M}_{L,\max}(G) \to \mathcal{M}_{L^2}(H^*). \tag{4.14.1}$$

4.3 Non-tube groups and stabilization of a maximal tube

In this section we study the moduli space of G-Higgs bundles with maximal Toledo invariant when G is of non-tube type.

Theorem 4.15. *Let G be a simple Hermitian group of non-tube type and let H be its maximal compact subgroup. Then, there are no stable G-Higgs bundles with maximal Toledo invariant. In fact, every polystable maximal G-Higgs bundle reduces to a stable $N_G(\mathfrak{g}_T)_0$-Higgs bundle, where $N_G(\mathfrak{g}_T)_0$ is the identity component of the normalizer of $\mathfrak{g}_T$ in G.*

Proof. Let $(E, \varphi = (\beta, \gamma))$ be a polystable G-Higgs bundle. Suppose that the Toledo invariant d is maximal. We assume that it is negative, $d = -r(2g-2)$ where $r = \mathrm{rk}(G/H)$, without loss of generality. Then, by Theorem 3.18, $\mathrm{rk}(\beta) = \mathrm{rk}(G/H) = r$. We define $P_r \subset H^{\mathbb{C}}$, σ and χ' as in the proof of Theorem 3.18. Since $\deg(E)(\sigma, \chi') = 0$ by 3.18, the polystability condition yields that $\varphi = (\beta, \gamma)$ belongs to $H^0(E_\sigma(\mathfrak{m}^0_\chi) \otimes K)$ and that the structure group reduces to $G_L = (L \cap H)\exp(\mathfrak{m}^0_\chi)$, where L is a Levi subgroup of P_r. From Lemma 2.70, we can take the Levi subgroup to be $N := N_{H^{\mathbb{C}}}(\mathfrak{h}^{\mathbb{C}}_T)_0$, which equals $(N_H(\mathfrak{h}_T)_0)^{\mathbb{C}}$.

Since $\mathfrak{m}^0_{\chi'} = \mathfrak{m}^{\mathbb{C}}_T$ by Lemma 2.71, the condition $\varphi = (\beta, \gamma) \in H^0(E_\sigma(\mathfrak{m}^0_{\chi'}) \otimes K)$ gives

$$\varphi \in H^0(E_\sigma(\mathfrak{m}^0_{\chi'}) \otimes K) = H^0(E_\sigma(\mathfrak{m}^{\mathbb{C}}_T) \otimes K).$$

From the discussion before Example 2.17, the maximal compact subgroup of $N_G(\mathfrak{g}_T)_0$ is $N_H(\mathfrak{h}_T)_0$ and the Cartan decomposition is $\mathfrak{n}_{\mathfrak{g}}(\mathfrak{g}_T) = \mathfrak{n}_{\mathfrak{h}}(\mathfrak{h}_T) + \mathfrak{m}_T$. Therefore, (E, φ) reduces to a $N_G(\mathfrak{g}_T)_0$-Higgs bundle in the sense of Definition 3.3. This new Higgs bundle is stable by Remark 3.6. $\square$

Remark 4.16. In the tube-type case, the argument of Theorem 4.15 does not work since the parabolic subgroup given by Theorem 3.18 is $P = H^{\mathbb{C}}$ and hence, there is no reduction of the structure group.

Theorem 4.15 was proved in [BGG06] for the classical groups. This general approach extends the result to quotients and coverings and to exceptional groups. A consequence is the following theorem.

Theorem 4.17. *Every maximal E_6^{-14}-Higgs bundle is strictly polystable and reduces to a stable* $\mathrm{Spin}_0(2,8) \times \mathrm{U}(1)$*-Higgs bundle and hence it is a product of a* $\mathrm{Spin}_0(2,8)$*-Higgs bundle and a line bundle. Moreover, the* $\mathrm{Spin}_0(2,8)$*-Higgs bundle is maximal.*

In the remainder of this section, we use the results of Section 2.6 to study the moduli space of maximal G-Higgs bundles and make an attempt to describing it as a fibration over a certain moduli space of holomorphic principal bundles with fibres isomorphic to maximal G_T-Higgs bundles. Recall from that section the notation $N = N_{H^{\mathbb{C}}}(H_T^{\mathbb{C}})_0$ and the definitions of $C_G = C_{H^{\mathbb{C}}}(\mathfrak{g}_T^{\mathbb{C}})$, $\Gamma_H = H_T^{\mathbb{C}} \cap Z(C_G)$ and $\Gamma_C = C_G' \cap Z(C_G)$. Assume from now on the hypothesis $C_G' \cap H_T^{\mathbb{C}} = 1$. As we have mentioned in Section 2.6, this hypothesis is seen to be true case by case, but there is no general proof for this fact. From Proposition 2.75, we have the exact sequence

$$1 \to N \to \frac{N}{H_T^{\mathbb{C}}} \times \frac{N}{C_G'} \to Q = \frac{Z(C_G)}{\Gamma\Gamma'} \to 1, \tag{4.17.1}$$

together with the diagram

$$\begin{tikzcd} N \arrow[r] \arrow[d] & N/H_T^{\mathbb{C}} \arrow[d] \\ N/C_G' \arrow[r] & Q, \end{tikzcd} \tag{4.17.2}$$

and the map $Q \to N/C_G'$ given by Lemma 2.78. We consider also the following diagram of cohomology pointed sets,

$$\begin{tikzcd} H^1(X,N) \arrow[r, "\pi"] \arrow[d, "\pi'"] & H^1(X, N/H_T^{\mathbb{C}}) \arrow[d, "c"] \\ H^1(X, N/C_G') \arrow[r, "c'"] & H^1(X,Q), \end{tikzcd} \tag{4.17.3}$$

where the groups N, N/C_G', $N/H_T^{\mathbb{C}}$ and Q denote the sheaves of functions in N, N/C_G', $N/H_T^{\mathbb{C}}$ and Q, respectively, Since Q is Abelian, $H^1(X,Q)$ is a group and we can take inverses of their elements. However, $H^1(X, N/H_T^{\mathbb{C}})$ and $H^1(X, N/C_G')$ are only pointed sets, in principle. We first prove the following lemma about $H^1(X,N)$, the set parameterizing principal N-bundles.

Lemma 4.18. *Assume that Γ_H is finite. The pointed set $H^1(X,N)$ fibers over $H^1(X, N/H_T^{\mathbb{C}})$. The fibres over the elements of $\{E \in H^1(X, N/H_T^{\mathbb{C}} \mid c(E)$ is a $o(\Gamma_H)$-th power $\}$ are all isomorphic to $H^1(X, H_T^{\mathbb{C}})$.*

Proof. We have that the set $H^1(X,N)$ fibres over $H^1(X,N/H_T^{\mathbb{C}})$. The fibre over a point $F_0 \in H^1(X,N/H_T^{\mathbb{C}})$ is isomorphic to

$$\{E \in H^1(X,N/C'_G) \mid c(F_0)c'(E) = \mathcal{O} \in H^1(X,Q)\}.$$

We check that it is indeed a fibration. Take any $\{g_{\alpha\beta}\} \in H^1(X,N/H_T^{\mathbb{C}})$ and any $\{h_{\alpha\beta}\} \in H^1(X,N/C'_G)$ over the fibre of $\{g_{\alpha\beta}\}$. We have that $\{g_{\alpha\beta}h_{\alpha\beta}\} = \mathcal{O} \in H^1(X,Q)$, and by the exact sequence 4.17.1 they determine uniquely a point $\{n_{\alpha\beta}\} \in H^1(X,N)$. The fibre $\pi^{-1}(\mathcal{O}) \subset H^1(X,N)$ corresponds to the principal $H_T^{\mathbb{C}}$-bundles, parameterized by $H^1(X,H_T^{\mathbb{C}})$.

We show how we may identify fibres using the diagram (4.17.3). Let $F_1 \in H^1(X,N/H_T^{\mathbb{C}})$. We regard the fibre over this element as:

$$M_{F_1} = \{E \in H^1(X,N/C'_G) \mid c(F_1)c'(E) = \mathcal{O}\}$$

Consider any element $T \in H^1(X,Q)$. For an element $E \in M_{F_1}$, we consider the fibre product $E \times_X T$. This is a principal $N/C'_G \times Q$-bundle. If $\{e_{\alpha\beta}\}$ and $\{t_{\alpha\beta}\}$ are the transition functions of E and T respectively, the transition functions of $E \times_X T$ are $\{(e_{\alpha\beta}, t_{\alpha\beta})\}$. The group Q acts both on E and T. Consider the bundle $E \times_X T/Q$. Its transition functions are the product $\{e_{\alpha\beta}t_{\alpha\beta}^{o(\Gamma_H)}\}$. They define a cocycle, and hence a bundle, because the image of Q is in the centre of N/C'_G. This new bundle satisfies $c'(E \times_X T) = c'(E)T^q$. Hence, two fibres M_{F_1} and M_{F_2} such that $c(F_1)c(F_2)^{-1} = T^q$ for some $T \in H^1(X,Q)$ are isomorphic. The isomorphism is given by the fibre product with T, and its inverse by the fibre product with T^{-1}. In particular, if $F_1 = \mathcal{O}$, we have the statement of the lemma. $\square$

Given a maximal G-Higgs bundle (E,φ), we get an $N_G(\mathfrak{g}_T)_0$-Higgs bundle by Theorem 4.15. This bundle consists of a principal N-bundle E_σ, given by an element of $H^1(X,N)$, together with a Higgs field $\varphi \in H^0(E_\sigma(\mathfrak{m}_T^{\mathbb{C}}) \otimes K)$.

We now indicate how the moduli space of polystable maximal G-Higgs bundles when G is of non-tube type is regarded as a fibration. This part may have some inaccuracies, but we include it in the thesis because it shows the role played by the geometric ingredients in the non-tube-type case.

Proposition 4.19. *Any maximal polystable G-Higgs bundle (E,φ) satisfies that $\pi(E_\sigma)$ has trivial or torsion topological class in $\pi_1(N/H_T^{\mathbb{C}})$.*

Proof. From any $E_\sigma \in H^1(X, N)$ and the diagram (4.17.3), we obtain a principal $N/H_T^{\mathbb{C}}$-bundle $\bar{E} = \pi(E_\sigma)$. Consider the character used in the proof of the Milnor-Wood inequality (Theorem 3.18) when $\mathrm{rk}(\beta)$ is maximal, $\chi = \chi_T - \chi_\Gamma$. We assume that the character lifts to the group. The proof of the inequality gives that $\deg(E_\sigma(\chi)) = 0$ and that the character χ is trivial in $H_T^{\mathbb{C}}$, so it descends to a character $\bar{\chi}$ of $N/H_T^{\mathbb{C}}$. This character is non-trivial in $N/H_T^{\mathbb{C}}$, so $\deg(\pi(E_\sigma)(\bar{\chi})$ is, up to a multiple, the projection of the topological class to the non-torsion part $\mathbb{Z}$. But $\deg(\pi(E_\sigma)(\bar{\chi})$ equals $\deg(E_\sigma(\chi)) = 0$, so the topological class is trivial or a torsion element. □

Remark 4.20. From $E_\sigma \in H^1(X, N)$ and the diagram (4.17.3) we also obtain a principal N/C'_G-bundle $\bar{E}' = \pi'(E_\sigma)$. The Toledo invariant of E is $\frac{1}{q_T}\deg(E(\chi_T^{q_T}))$. We define the character $\chi_T^{q_T}$ in N by restriction, and in N/C'_G by projection, since $C'_G \subset [H^{\mathbb{C}}, H^{\mathbb{C}}]$. We have that

$$\deg(E(\chi_T^{q_T})) = \deg(E_\sigma(\chi_T^{q_T})) = \deg(\bar{E}'(\chi_T^{q_T})), \tag{4.20.1}$$

and these degrees are maximal. This is indicating maximality for the bundle $\bar{E}'$ although the group N/C'_G is not simple, as we will see in Question 4.21. Maximality makes also sense for general reductive groups of Hermitian type, although we have defined the concept only for simple groups.

We use the fibration of principal bundles of Lemma 4.18 and Proposition 4.19 to suggest how the moduli space $\mathcal{M}_{max}(G)$ is regarded as a fibration.

Question 4.21. *Let G be a group of non-tube type such that $C'_G \cap H_T^{\mathbb{C}} = 1$ and Γ_H is finite. Does the moduli space of maximal G-Higgs bundles for a non-tube group G, $\mathcal{M}_{max}(G)$, fibre over a subvariety of the moduli space of principal $N/H_T^{\mathbb{C}}$-bundles with trivial or torsion topological class in $\pi_1(N/H_T^{\mathbb{C}})$, $M_0(N/H_T^{\mathbb{C}})$? Is the fibre over the elements of trivial topological class isomorphic to the moduli space of maximal G_T-Higgs bundles, $\mathcal{M}_{max}(G_T)$?*

A possible argument to give a positive answer to this question may be the following. By Theorem 4.15 and Proposition 4.19, we have a correspondence between $\mathcal{M}_{max}(G)$ and

$$\mathcal{M}_{max}(N_G(\mathfrak{g}_T)_0) := \{E \in \mathcal{M}(N_G(\mathfrak{g}_T)_0) \mid {}^1\!/_{q_T} \deg(E(\chi_T^{q_T})) \text{ maximal}\}.$$

Any $(E, \varphi) \in \mathcal{M}_{max}(N_G(\mathfrak{g}_T)_0)$ consists of $E \in H^1(X, N)$ and $\varphi \in H^0(E(\mathfrak{m}_T^{\mathbb{C}}) \otimes K)$. The fibration of Lemma 4.18 gives the principal bundles $\bar{E} \in H^1(X, N/H_T^{\mathbb{C}})$ and

$F \in H^1(X, N/C'_G)$. The Higgs field is attached to the principal bundle F, since $N/H_T^{\mathbb{C}}$ acts trivially on $\mathfrak{m}_T^{\mathbb{C}}$. Thus, we obtain a $N_G(G_T)_0/C'_G(G_T)_0$-Higgs bundle $(\bar{E}, \varphi)$ and a holomorphic principal $N/H_T^{\mathbb{C}}$-bundle. We have therefore a projection to $M(N/H_T^{\mathbb{C}})$, where the fibres consist of $N_G(G_T)_0/C'_G(G_T)_0$-Higgs bundles. The isomorphisms between fibres described in Lemma 4.18. By Proposition 4.19, the moduli space projects into a subvariety of the moduli space $M_{0,tor}(N/H_T^{\mathbb{C}})$ of holomorphic principal $N/H_T^{\mathbb{C}}$-bundles with trivial or torsion topological class. For the elements of the base with trivial topological class there exist $o(\Gamma_H)$-roots, and Lemma 4.18 actually gives an isomorphism between the fibres over these elements. We look at the fibre over $F = \mathcal{O}$ as a model. In this case, a bundle E_σ is regarded as an element in $H^1(X, H_T^{\mathbb{C}})$. Since $\mathfrak{m}_T^{\mathbb{C}}$ is the isotropy representation of $H_T^{\mathbb{C}}$, the pair $(E_\sigma, \varphi \in H^0(E_\sigma(\mathfrak{m}_T^{\mathbb{C}}) \otimes K)$ is a G_T-Higgs bundle. The degree of $E_\sigma(\chi_T)$ equals the rank of both the symmetric space G/H and its maximal tube subdomain. This degree equals the Toledo invariant, so this fibre consist of bundles with maximal Toledo invariant. It would remain to check the correspondence of the stability conditions of the bundles involved to get a positive answer to Question 4.21

Remark 4.22. The hypothesis that Γ_H is finite seems to be satisfied for all the cases, but no classification-independent proof has been already provided.

A casewise positive answer to Question 4.21 is given in [BGG06] for the classical groups. We sketch the proof for the group $\mathrm{SU}(p, q)$ as it is the motivation of the general approach and mention the case of $\mathrm{SO}^*(4m+2)$ for its similarity with the exceptional case.

An $\mathrm{SU}(p, q)$-Higgs bundle consists of a principal $S(\mathrm{GL}(p, \mathbb{C}) \times \mathrm{GL}(q, \mathbb{C}))$-bundle together with the Higgs field. As mentioned in Example 3.14, this principal bundle can be represented by two vector bundles V and W of rank p and q respectively such that $\det V \det W = \mathcal{O}$, and the Higgs field has two components: $\beta : W \to V \otimes K$ and $\gamma : V \to W \otimes K$. For maximal Toledo invariant, γ is an isomorphism onto its image $\gamma : V \cong W' \otimes K$, where $W' = \mathrm{im}(\gamma) \otimes K^{-1}$. We define W'' as W/W' to get

$$(V, W, \beta, \gamma) \cong (V, W', \beta, \gamma) \oplus (0, W'', 0, 0)$$

together with the condition $\det V \otimes \det W' \otimes \det W'' \cong \mathcal{O}$.

This is what Example 2.76 describes when we pass from the matrices A, B, C to the bundles V, W', W'', and it is the motivation to use the sequence (4.17.1) to get this fibration in general.

The case of $\mathrm{SO}^*(4m+2)$ is simpler because $N/H_T^{\mathbb{C}}$ is a circle. One proves that $\mathcal{M}_{max}(\mathrm{SO}^*(4m+2))$ is homeomorphic to the product of $\mathcal{M}_{max}(\mathrm{SO}^*(4m))$ and the Jacobian of X. The situation for the exceptional group is very likely to be the same.

Question 4.23. *Is the moduli space $\mathcal{M}_{max}(E_6^{-14})$ is homeomorphic to the product $\mathcal{M}_{max}(\mathrm{Spin}_0(2,8)) \times J(X)$, where $J(X)$ is the Jacobian of X?*

These results describing a fibration may be helpful to study the topology of the moduli space of maximal Higgs bundles. Since $M_0(N/H_T^{\mathbb{C}})$ is always connected, we have that the connectedness of $\mathcal{M}_{max}(G_T)$ would imply the connectedness $\mathcal{M}_{max}(G)$. This has been used in [BGG06] to prove that the moduli space $\mathcal{M}_{max}(\mathrm{SO}^*(4m+2))$ is connected. Nonetheless, if $\mathcal{M}_{max}(G_T)$ is not connected, we cannot say anything about the connectedness of $\mathcal{M}_{max}(G)$. In fact, in [BGG06] it is proved that the moduli space $\mathcal{M}_{max}(\mathrm{SU}(p,q))$ is connected although $\mathcal{M}_{max}(\mathrm{SU}(p,p))$ has 2^{2g} connected components, when g is the genus of X. In the case of E_6^{-14}, we have that $\mathcal{M}_{max}(\mathrm{Spin}_0(2,8))$ is not connected, so a further study is needed to know the number of connected components.

Chapter 5

Surface group representations

Let G be a semisimple Lie group, and $\pi_1 X$ be the fundamental group of a smooth compact surface X of genus g, which is finitely presented as

$$\pi_1 X = \langle a_1, b_1, \ldots, a_g, b_g \mid \prod_{j=1}^{g} [a_j, b_j] = 1 \rangle.$$

Every representation $\rho : \pi_1 X \to G$ is identified with an element of G^{2g} whose $2g$ components satisfy the commutation relation. The space of representations $\mathrm{Hom}(\pi_1 X, G)$ is then regarded as a subvariety of G^{2g}. The group G acts by conjugation on $\mathrm{Hom}(\pi_1 X, G)$, but the orbit space $\mathrm{Hom}(\pi_1 X, G)/G$ does not have in general the structure of Hausdorff space. To solve this, one considers the subset $\mathrm{Hom}^+(\pi_1 X, G)$ consisting of reductive representations. These are representations which are completely reducible when composed with the adjoint representation, or, when G is algebraic, which satisfy that the closure of $\rho(\pi_1 X)$ is Zariski dense. The moduli space of representations, or character variety, is defined as

$$\mathcal{R}(\pi_1 X, G) = \mathrm{Hom}^+(\pi_1 X, G)/G.$$

This moduli space is in general a real analytic variety, and when G is algebraic, has an algebraic structure.

5.1 Correspondence with Higgs bundles

In order to illustrate how the definition of G-Higgs bundle given in Section 3.1 arises naturally from the correspondence of surface group representations with Higgs bundles, we start by considering the group $G = \mathrm{SU}(r)$.

We first review the correspondence between representations of $\pi_1 X$ into the special unitary group and flat connections. The universal cover $\tilde{X}$ of X is a principal $\pi_1 X$-bundle over X by the action of $\pi_1 X$ on $\tilde{X}$ by deck transformations. Since $\pi_1 X$ is discrete, locally constant transition functions can be chosen to give $\tilde{X} \to X$ the structure of flat bundle. Given any representation $\rho : \pi_1 X \to \mathrm{SU}(r)$, define the associated bundle $E_\rho = E \times_{\pi_1 X} \mathrm{SU}(r)$. This bundle has trivial determinant and carries a flat structure, and therefore a connection with zero curvature. Conversely, any complex vector bundle of rank r and trivial determinant with Hermitian structure h and flat h-connection D defines a representation $\rho : \pi_1 X \to \mathrm{SU}(r)$ such that $E = E_\rho$. This representation is defined as follows. Take a loop in X. The horizontal lifting defines an isomorphism of the fibre. This isomorphism lies in the monodromy group, which is inside $\mathrm{SU}(r)$ due to the Hermitian structure and the fact that the determinant is trivial. The flatness of the connection makes this map depend only on the homotopy class of the loop, and thus defines a representation $\pi_1 X \to \mathrm{SU}(r)$. In 1965, Narasimhan and Seshadri ([NS65]) proved that the irreducible representations are in correspondence with the stable holomorphic vector bundles of rank r and trivial determinant. A vector bundle E is said to be stable if for every proper subbundle $F \subset E$ one has $\frac{\operatorname{rk} F}{\deg F} < \frac{\operatorname{rk} E}{\deg E}$, where the quotient $\frac{\operatorname{rk} F}{\deg F}$ is called the slope of the vector bundle F. This notion of slope-stability had been used by Mumford ([Mum63]) to give the set of isomorphism classes of stable bundles the structure of an algebraic variety.

We replace now the special unitary group $\mathrm{SU}(r)$ by an arbitrary semisimple Lie group G and state a similar correspondence. As well as for $\mathrm{SU}(r)$, there is a correspondence between flat principal G-bundles over X and representations of $\pi_1 X$ into G. Let $B \in \Omega^1(E, \mathfrak{g})$ be a connection in the principal G-bundle E. Let $h : E \to E/H$, or equivalently $i : E_H \to E$, be a reduction of the structure group of E from G to a maximal compact subgroup H. Note that in the case of $\mathrm{SU}(n) \subset \mathrm{SL}(n, \mathbb{C})$, this reduction is a Hermitian metric on the complex vector bundle. Consider $i^* B \in \Omega^1(X, E_H(\mathfrak{g}))$. The Cartan decomposition $\mathfrak{g} = \mathfrak{h} + \mathfrak{m}$ described in Section 2.1, which generalizes the decomposition of a matrix into Hermitian and skew-Hermitian part, gives $i^* B = A + \theta$ with $A \in \Omega^1(E_H, \mathfrak{h})$ a connection on E_H and $\theta \in \Omega^1(X, E_H(\mathfrak{m}))$. The field θ measures the discrepancy of the connection to be an H-connection. A similar correspondence to the theorem of Narasimhan and Seshadri about representations and holomorphic bundles is stated by endowing the holomorphic bundles with extra structure coming from θ. As we will see, the resulting object is a G-Higgs bundle, as defined in

Section 3.1. We outline the proof of this correspondence not only to show how G-Higgs bundles arise, but also to show the interplay between algebraic, topological and differential objects. The proof of the correspondence is based on the relation with Yang-Mills theories, as shown in [AB83] (before the introduction of Higgs bundles), [Hit87] and [Sim94a], and makes use of the Hitchin-Kobayashi correspondence mentioned in Section 3.2.

A theorem of Donaldson for $G = \mathrm{PSL}(2,\mathbb{C})$ ([Don87]) and Corlette for an arbitrary semisimple Lie group G ([Cor88]) states that a representation $\rho : \pi_1 X \to G$ is reductive if and only if $E_\rho = \widetilde{X} \times_\rho G$ admits a harmonic metric with respect to any conformal structure in X, or equivalently, a complex structure, making X into a Riemann surface.

Fixing a smooth principal G-bundle E_G, the above correspondence between representations of G and G-connections is expressed in terms of moduli spaces as

$$\mathrm{Hom}^+(\pi_1 X, G)/G \cong \{\text{Reductive flat } G\text{-connections } B \ \mid F(B) = 0\}/\mathcal{G}_G,$$

where $\mathcal{G}_G$ is the gauge group of E_G, and the reductive connections are those whose associated representation is reductive.

Given a reduction h of E_G to E_H, we have the decomposition $i^*B = A + \theta$ mentioned above. The condition that the connection is flat, $dB + \frac{1}{2}[B,B] = 0$, written in terms of A and θ becomes,

$$d\theta + [A,\theta] = 0, \quad \text{or equivalently} \quad \left.\begin{array}{r} F(A) + \frac{1}{2}[\theta,\theta] = 0 \\ d_A\theta = 0 \end{array}\right\}, \tag{5.0.1}$$

where d_A is the covariant derivative $d_A : \Omega^0(X, E_H(V)) \to \Omega^1(X, E_H(V))$ associated to the connection A. Moreover, we have the condition coming from the harmonicity of the metric, which corresponds to

$$d_A^*\theta = 0. \tag{5.0.2}$$

The gauge group $\mathcal{G}_H$ of E_H acts on the moduli space of solutions of equations (5.0.1) and (5.0.2), and for the moduli spaces one has the equivalence

$$\begin{aligned} &\{\text{Reductive flat } G\text{-connections } B \ \mid F(B) = 0\}/\mathcal{G}_G \\ &\quad \cong \{(A,\theta) \text{ satisfying (5.0.1) and (5.0.2) }\}/\mathcal{G}_H. \end{aligned}$$

A connection A in E_H induces a holomorphic structure in the holomorphic principal $H^{\mathbb{C}}$- bundle $E_H \times_H H^{\mathbb{C}}$. The associated covariant derivative on sections decomposes into $(1,0)$ and $(0,1)$ parts as $d_A = \partial_A + \bar{\partial}_A$. Recall that the holomorphic sections are given by $\mathrm{Ker}(\bar{\partial}_A)$. From $\theta \in \Omega^1(X, E_H(\mathfrak{m}))$, the inclusion $E_H(\mathfrak{m}) \subset E_H(\mathfrak{m}^{\mathbb{C}}) = E_{H^{\mathbb{C}}}(\mathfrak{m}^{\mathbb{C}})$, and the equivalence $T^*X \cong T^*X^{(1,0)}$, we define $\varphi \in \Omega^{1,0}(X, E_{H^{\mathbb{C}}}(\mathfrak{m}^{\mathbb{C}}))$. The equations (5.0.1) and (5.0.2) written in terms of A and φ are

$$\left.\begin{aligned} F(A) - [\varphi, \tau_h(\varphi)] = 0 \\ \bar{\partial}_A \varphi = 0 \end{aligned}\right\}, \tag{5.0.3}$$

where τ_h is defined as in Theorem 3.10. The equivalence of the equations gives

$$\{(A, \theta) \text{ satisfying (5.0.1) and (5.0.2) }\}/\mathcal{G}_H \cong \{(A, \varphi) \text{ satisfying (5.0.3) }\}/\mathcal{G}_H.$$

Equations (5.0.3) are known as the Hitchin equations. From the condition $\bar{\partial}_A \varphi = 0$, one has that φ is holomorphic, $\varphi \in H^0(X, E_{H^{\mathbb{C}}}(\mathfrak{m}^{\mathbb{C}}) \otimes K)$, i.e., we obtain a Higgs field, and thus $(E_{H^{\mathbb{C}}}, \varphi)$ is a G-Higgs bundle. By Theorem 3.10 for $\alpha = 0$, this resulting Higgs bundle $(E_{H^{\mathbb{C}}}, \varphi)$ satisfying the Hitchin equations is polystable (i.e., α-polystability for $\alpha = 0$ in Definition 3.5). However, for a polystable G-Higgs bundle $(E_{H^{\mathbb{C}}}, \varphi)$, the corresponding connection A and field θ do not necessarily satisfy the Hitchin equations. Nonetheless, one has that in the orbit of any polystable G-Higgs bundle by the action of $\mathcal{G}_{H^{\mathbb{C}}}$ (the gauge group of $E_{H^{\mathbb{C}}}$), there is another polystable G-Higgs bundle such that the connection and field θ satisfy the equations. The connection and θ are determined up to the action of $\mathcal{G}_H$. This gives the last equivalence of moduli spaces

$$\{(A, \varphi) \text{ satisfying (5.0.3) }\}/\mathcal{G}_H \cong \mathcal{M}(G),$$

where the moduli space $\mathcal{M}(G)$ parameterizes equivalence classes polystable G-Higgs bundles under the action of $\mathcal{G}_{H^{\mathbb{C}}}$.

As in Section 3.5 the moduli space of representations can be sliced using the topological classification of G-bundles. From the exact sequence $1 \to \pi_1 G \to \tilde{G} \to G \to 1$ we get the long exact sequence in cohomology and the connection homomorphism $H^1(X, G) \xrightarrow{\delta} H^2(X, \pi_1 G)$, where the domain parameterizes principal G-bundles and the target is isomorphic to $\pi_1 G \cong \pi_1 H$ since H is a deformation retract of G. This map thus associates a topological class in $\pi_1 H$ to any G-bundle, and in particular to any representation $\rho : \pi_1 X \to G$ by means of E_ρ. The moduli space of representations splits into $\mathcal{R}(\pi_1 X, G) = \bigcup_{d \in \pi_1 H} \mathcal{M}_d(\pi_1 X, G)$, where

$$\mathcal{M}_d(\pi_1 X, G) = \{[\rho] \in \mathcal{R}(\pi_1 X, G) \mid \delta(E_\rho) = d\}.$$

Theorem 5.1. *There is an isomorphism as real-analytic varieties between the moduli space of representations into G with topological class $d \in \pi_1 H$ and the moduli space of G-Higgs bundles with topological class $d \in \pi_1 H$,*

$$\mathcal{R}_d(\pi_1 X, G) \cong \mathcal{M}_d(G). \tag{5.1.1}$$

If the moduli space of Higgs bundles is sliced using the Toledo invariant, and not the topological class, one gets a correspondence with the moduli space of representations sliced using the Toledo invariant. Namely, to a representation $\rho : \pi_1 X \to G$, one associates the Toledo invariant of the vector bundle E_ρ.

5.2 Equivalent definitions of the Toledo invariant

In the present work we define the Toledo invariant as the degree of the line bundle associated to the Toledo character. In Section 3.5 we related this definition to the one given in [BGG06] as a topological class, which corresponds in turn to the Toledo invariant for representations defined in Section 5.1. In this section we compare these definitions with the definition coming from the study of surface group representations.

The Toledo invariant of a representation $\rho : \pi_1 X \to G$ was originally defined by Toledo in [Tol89] for $\mathrm{PU}(1,n)$. This definitions works for any other Hermitian group as follows. The Hermitian space G/H carries a unique Hermitian (normalized) metric of minimal holomorphic sectional curvature -1, the Bergmann metric. Consider the Kähler form ω of this metric. The representation $\rho : \pi_1 X \to G$ determines a ρ-equivariant map $f : \widetilde{X} \to G/H$, known as the developing map. The pullback $f^*\omega$ defines a $\pi_1 X$-invariant form on $\widetilde{X}$ and hence descends to a form on X which we denote also by $f^*\omega$. The Toledo invariant is then defined as

$$T(\rho) = \int_X f^*\omega.$$

This definition was extended by Burger, Iozzi and Wienhard by using methods of bounded cohomology, as is surveyed in [BIW10b]. The Kähler form of the Bergmann metric defines a continuous cohomology class which is always bounded, $\kappa_G \in H^2_{cb}(G, \mathbb{R})$. Given the representation $\rho : \pi_1 X \to G$, consider the form $\rho^*(\kappa_G)$. The Toledo invariant is then defined as the evaluation of this form on the fundamental class $[X]$ of the surface, which is equivalent to the integration process above. The novelty of this definition is that it can be extended to surfaces with boundary. The continuous cohomology class determined by the Kähler form is not necessarily bounded, but a bounded Kähler class is obtained by considering the isomorphism between bounded continuous and continuous cohomology.

Alternatively, this characteristic number is obtained, up to a multiple, as the Chern class of a complex line bundle over G/H. When G/H is irreducible, the curvature form is a multiple of the normalized Kähler form defined above. We can give such a complex line bundle from the representation in the following way. The group G is a principal H-bundle over G/H. Assume in this Section that the Toledo character lifts to the group, $\chi_T : H^{\mathbb{C}} \to \mathbb{C}^*$, for the sake of simplicity. We get a representation $\chi_T : H \to U(1)$ and thus an associated line bundle $L := G(\chi_T)$ over G/H. From the ρ-equivariant map $f : \widetilde{X} \to G/H$, we obtain the bundle f^*L over $\widetilde{X}$. The quotient by the action of $\pi_1 X$ is a line bundle over X, $f^*L/\pi_1 X$.

In order to connect this approach to our definition, we first define a $H^{\mathbb{C}}$-bundle from a representation ρ, using the correspondence described in Section 5.1. We then show that the Toledo invariant of this bundle equals the Toledo invariant of the representation. Starting with the principal $\pi_1 X$-bundle $\widetilde{X} \to X$ and the representation $\rho : \pi_1 X \to G$, we obtain a principal G-bundle, $\widetilde{X}(\rho)$. From the ρ-equivariant map $f : \widetilde{X} \to G/H$ we define a reduction of $\widetilde{X}(\rho)$ from G to H by the map $\widetilde{X}(\rho) \to G/H$ given by $[x, g] \mapsto f(x)g$, which is well-defined by the ρ-equivariance of f. The resulting reduction is an H-bundle E_H. We complexify it to obtain $E = E_H \times H^{\mathbb{C}}$, to which we associate a line bundle using the Toledo character χ_T, $E(\chi_T)$. In fact, this line bundle $E(\chi_T)$ is isomorphic to the line bundle $f^*L/\pi_1 X$ obtained above, since we have the same $\pi_1 X$ and H-equivariances.

From Lemma 2.41, $\chi_T([Y, Z])$ defines a Kähler form $\omega(Y, Z)$ in G/H.

This form equals, up to scaling, the Kähler form with minimal holomorphic sectional curvature -1.

Lemma 5.2. *The Kähler form $\omega(Y, Z) = \chi_T([Y, Z])$ given by Lemma 2.41 has minimal holomorphic sectional curvature -1.*

Proof. We use the formula for the holomorphic sectional curvature,

$$\kappa(X) = \frac{g(R(X, J_0X)J_0X, X)}{g(X, X)g(J_0X, J_0X) - g(X, J_0X)^2}.$$

The curvature of the connection in a non-compact symmetric space is given by $R(X, Y)Z = [[X, Y], Z]$. Since the space is Kähler, the metric is given by $g(X, Y) = \omega(X, J_0Y) = \chi_T([X, J_0Y])$, where J_0 is the almost complex structure on $\mathfrak{m}$. Consider the elements $\{e_\alpha, e_{-\alpha}, h_\alpha\}_{\alpha \in \Delta_Q^+}$ introduced in Section 2.1. Take the basis of $\mathfrak{m}$ consisting of $\{x_\alpha = e_\alpha + e_{-\alpha}\}_{\alpha \in \Delta_Q^+}$. We have that $J_0 x_\alpha = y_\alpha = i(e_\alpha - e_{-\alpha})$. One has that $[x_\alpha, y_\alpha] = -2ih_\alpha$, $[-2ih_\alpha, y_\alpha] = 8e_\alpha$, $[8e_\alpha, x_\alpha] = 8h_\alpha$ and hence,

$$\kappa(x_\alpha) = \frac{\chi_T([[[x_\alpha, y_\alpha], y_\alpha], y_\alpha])}{\chi_T([x_\alpha, y_\alpha])^2 - \chi_T([x_\alpha, -x_\alpha])^2} = \frac{\chi_T(8h_\alpha)}{\chi_T(-2ih_\alpha)^2} = -\frac{2}{\chi_T(h_\alpha)}.$$

Since $\chi_T(h_\alpha) = 2$ by Lemmas 2.33 and 2.34, we have that the holomorphic sectional curvature is -1. □

Yet another approach to the Toledo invariant is possible using rotation numbers as shown in [BIW10a].

5.3 Milnor-Wood inequality and rigidity of maximal representations

We state versions of the main results of this thesis for surface group representations, by taking advantage of the correspondence between Higgs bundles and surface group representations, and the equivalence of the definitions of the Toledo invariant.

From the Milnor-Wood inequality for G-Higgs bundles (Theorem 3.18), we obtain the Milnor-Wood inequality for representations.

Theorem 5.3. *Let $\rho : \pi_1 X \to G$ be a maximal representation of the fundamental group of a Riemann surface X into a semisimple Hermitian Lie group G. The Toledo invariant d of ρ satisfies*

$$-\operatorname{rk}(G/H)(2g-2) \leq d \leq \operatorname{rk}(G/H)(2g-2).$$

The first version of this inequality was proved by Milnor ([Mil58]) in the case of $\mathrm{SL}(2,\mathbb{R}) \cong \mathrm{Sp}(2,\mathbb{R})$, where the Toledo invariant coincides with the Euler class of the $\mathrm{SL}(2,\mathbb{R})$-bundle. This bounding was generalized by Wood in [Woo71]. Bounds on characteristic classes were also obtained by Dupont ([Dup79]) for classical groups and by Turaev [Tur84] for the symplectic group. Domic and Toledo provided a proof for the groups $\mathrm{SU}(p,q)$, $\mathrm{SO}^*(2n)$ and $\mathrm{Sp}(2n,\mathbb{R})$ in [DT87]. These results were generalized, using reproducing kernels, by Clerc and Orsted in [CØ03]. A general proof has been also given using methods of bounded cohomology by Burger, Iozzi, and Wienhard in [BILW05]. For a different approach to the Milnor-Wood inequality for Higgs bundles, one may look also at [HO11]. The study of maximal representations has attracted much interest because of its geometric significance. In the case of $\mathrm{SL}(2,R)$, Goldman ([Gol80]) proved that there are 2^{2g} maximal components in the moduli space of representations, which can be identified with the Teichmüller space, and they consist entirely of discrete and faithful representations. Moreover, the work of Goldman ([Gol82]) deals with a $(n:1)$-coverings of $\mathrm{PSL}(2,\mathbb{R})$, so it is somehow concerned with the same issues of Remark 2.8 which lead to the use of constants associated to the group, as $o(e^{2\pi J})$ and $l = Z_0^{\mathbb{C}} \cap [H^{\mathbb{C}}, H^{\mathbb{C}}]$.

Using methods of bounded cohomology, Burger, Labourie, Iozzi and Wienhard have proved in general that the maximal components for Hermitian groups consist entirely of discrete and faithful representations. An interesting result coming from this approach is that maximal representations are necessarily reductive, so the hypothesis of reductivity for Higgs bundles is trivially satisfied in the maximal case.

For tube-type groups, the Cayley correspondence $\mathcal{M}_{max}(G) \cong \mathcal{M}_{K^2}(H^*)$ proved in Theorem 4.7, shows the rigidity of maximal objects. The corresponding K^2-twisted H^*-pair has a reduction group smaller than the initial G-Higgs bundle. Although it is not a reduction of the structure group, the dimension of the new group H^* equals the dimension of a maximal compact subgroup H of G. Furthermore, this correspondence reveals new invariants for maximal G-Higgs bundles and surface group representation into G. These are the invariants coming from the group H^*. For example, when $G = \mathrm{Sp}(2n, \mathbb{R})$, we have that $H^* = \mathrm{GL}(n, \mathbb{R})$ with $H' = \mathrm{O}(n)$ as a maximal compact subgroup. To a $\mathrm{Sp}(2n, \mathbb{R})$-Higgs bundle we can thus attach the first and second Stiefel-Whitney classes of the corresponding $\mathrm{GL}(n, \mathbb{R})$-Higgs pair via the Cayley correspondence, $w_1 \in H^1(X, \mathbb{Z}/2)$, $w_2 \in H^2(X, \mathbb{Z}/2)$. In the moduli space $\mathcal{M}_{max}(\mathrm{Sp}(2n, \mathbb{R})$ there are 2^{2g} Hitchin components $\mathcal{M}_L^H$ indexed by square roots L of K. Define $\mathcal{M}_{w_1,w_2}(\mathrm{Sp}(2n, \mathbb{R}))$ as the subspace of maximal $\mathrm{Sp}(2n, \mathbb{R})$ not belonging to any Hitchin component. For $n \geq 3$, these are all the connected components:

$$\mathcal{M}_{max}(\mathrm{Sp}(2n, \mathbb{R})) = \bigcup_{w_1,w_2} \mathcal{M}_{w_1,w_2}(\mathrm{Sp}(2n, \mathbb{R})) \cup \bigcup_{L^2=K} \mathcal{M}_L^H(\mathrm{Sp}(2n, \mathbb{R})),$$

as stated in [GGM08]. When $n = 2$, yet more invariants may appear, as shown in [BGG09]. When $w_1 = 0$, $w_2 \in H^2(X, \mathbb{Z}/2)$ lifts to a class $c \in H^2(X, \mathbb{Z})$ and one defines the space $\mathcal{M}_{0,c}(\mathrm{Sp}(4, \mathbb{R}))$ of maximal $\mathrm{Sp}(4, \mathbb{R})$ not belonging to the Hitchin components, satisfying $w_1 = 0$ and w_2 lifting to $c \in H^2(X, \mathbb{Z})$. For $\mathrm{Sp}(4, \mathbb{R})$ the decomposition in connected components is given by

$$\mathcal{M}_{max}(\mathrm{Sp}(4, \mathbb{R})) = \bigcup_{w_1 \neq 0, w_2} \mathcal{M}_{w_1,w_2}(\mathrm{Sp}(4, \mathbb{R})) \cup \bigcup_{0 \leq c < 2g-2} \mathcal{M}_{0,c}(\mathrm{Sp}(4, \mathbb{R})) \cup \bigcup \mathcal{M}_L^H.$$

A concise reference for these results is [Got11].

These invariants have appeared from the point of view of representations in the work of Guichard and Wienhard. In Theorems 3 and 4 of [GW09] they define analogues of w_1, w_2 and c above for $(\mathrm{Sp}(2n, \mathbb{R}), \mathrm{GL}(n, \mathbb{R}))$-Anosov representations, a concept which include maximal representations. The study of the Hitchin component for $\mathrm{SL}(n, \mathbb{R})$ by Labourie yielded the concept of Anosov representation ([Lab06]) which has been generalized later.

For non-tube-type groups, we obtain the following result from Theorem 4.15.

Theorem 5.4. *Let $\rho : \pi_1 X \to G$ be a maximal representation of the fundamental group of a Riemann surface X into a semisimple Hermitian Lie group of non-tube type G. Then, the image of ρ is contained in $N_G(\mathfrak{g}_T)_0$, where $\mathfrak{g}_T$ is the subalgebra of $\mathfrak{g}$ corresponding to a maximal tube type subdomain G_T/H_T of G/H.*

In particular, for the exceptional case we have

Theorem 5.5. *The image of any maximal representation $\rho : \pi_1 X \to E_6^{-14}$ is contained in* $\mathrm{Spin}_0(2,8) \times \mathrm{U}(1)$.

The rigidity of representations of maximal surface group representations was detected by Toledo [Tol89] for the group $\mathrm{PU}(1,q)$, showing that such a maximal representation stabilizes a complex geodesic. This complex geodesic corresponds, for this group, to a maximal tube-type subdomain, so the image is contained in the normalizer of the tube-type subdomain.

This was generalized by Hernández in [Her91] for the group $\mathrm{PSU}(2,q)$. The generalization to $\mathrm{PU}(p,q)$ was obtained using Higgs bundles techniques by Bradlow, García-Prada and Gothen in [BGG01]. This work has been followed by the study of $\mathrm{U}(p,q)$ in [BGG03], and the work about $\mathrm{SO}^*(4n+2)$ ([BGG12]).

A different and general approach using bounded cohomology has been used by Burger, Iozzi, Labourie, Wienhard, as it is explained in [BIW10b]. In fact a version of Theorem 5.4 with the same notation appears in a preliminary version ([BIW06]).

5.4 Future directions

The relation between Higgs bundles and surface group representations suggests many questions that may be of interest. We gather here some of them.

1. From the Milnor-Wood inequality for G-Higgs bundles (Theorem 3.18), we obtain a Milnor-Wood type inequality for representations. Using the correspondence with Higgs bundles, we may associate to a representation $\rho : \pi_1 X \to G$ a pair of integers r_β, r_γ corresponding to the ranks of the fields β and γ of the associated G-Higgs bundle (E, β, γ) to ρ. This would give the following result.

   Theorem 5.6. *Let $\rho : \pi_1 X \to G$ be a maximal representation of the fundamental group of a Riemann surface X into a semisimple Hermitian Lie group G. The Toledo invariant d of ρ satisfies*

   $$-r_\beta(2g-2) \leq d \leq r_\gamma(2g-2).$$

What is the meaning of r_β and r_γ from the point of view of surface group representations? Can the Cayley correspondence be stated in terms of surface group representations? How is this related to boundary maps to the Shilov boundary arising from maximal representations, as described in [BIW10b]? How is this related to the work on weakly maximal representations announced in [BBH$^+$11]?

2. Maximal representations are in particular H'-Anosov. From Anosov representations $\pi_1 X \to G$, geometric structures are explicitly constructed in [GW11]. Namely, open subsets of compact G-spaces, on which $\pi_1 X$ acts properly discontinuously and with compact quotient.

   Is there any notion of Anosov for Higgs bundles? How are these structures reflected in Higgs bundles?

3. The notion of Toledo invariant is defined in [BIW10b] for an oriented compact surface with boundary, by using bounded cohomology classes. Surfaces with boundary include punctured Riemann surfaces, for which the concept of parabolic Higgs bundle was introduced by Simpson ([Sim88],[Sim90]) and has been widely studied. A Toledo invariant has been defined in the context of parabolic Higgs bundles as in [GLM09] for $\mathrm{U}(p,q)$. It would be very interesting to establish in the generality of this thesis the parabolic situation. Steps in this direction are taken in [BGM10].

4. The concept of α-polystability was introduced in Section 3.1. When $\alpha = 0$, we obtain the correspondence with surface group representations. Are there any geometrical objects corresponding naturally to α-polystable Higgs bundles when $\alpha \neq 0$? In that case, a Milnor-Wood type inequality is already provided by Theorem 3.18, is there any rigidity associated to the maximal corresponding object?

Appendix A

Examples

$\mathfrak{su}(p,q)$

$$\mathfrak{su}(p,q) = \left\{ X \in \mathfrak{sl}(p+q,\mathbb{C}) \mid X^* \left(\begin{array}{c|c} \mathrm{Id}_p & \\ \hline & -\mathrm{Id}_q \end{array}\right) + \left(\begin{array}{c|c} \mathrm{Id}_p & \\ \hline & -\mathrm{Id}_q \end{array}\right) X = 0 \right\} =$$

$$= \left\{ \left(\begin{array}{c|c} A & B \\ \hline B^* & D \end{array}\right) \mid \begin{array}{l} A \in \mathfrak{u}(p), D \in \mathfrak{u}(p), B \in \mathrm{Hom}(\mathbb{C}^p,\mathbb{C}^q) \\ \mathrm{tr}\, A + \mathrm{tr}\, D = 0 \end{array} \right\}$$

$$\mathfrak{h} = \left\{ \left(\begin{array}{c|c} A & 0 \\ \hline 0 & D \end{array}\right) \mid A, D \in \mathfrak{u}(n), \mathrm{tr}\, A + \mathrm{tr}\, D = 0 \right\} = \mathfrak{s}(\mathfrak{u}(p) \oplus \mathfrak{u}(q))$$

$$\mathfrak{m} = \left\{ \left(\begin{array}{c|c} 0 & B \\ \hline B^* & 0 \end{array}\right) \mid B \in \mathrm{Hom}(\mathbb{C}^p,\mathbb{C}^q) \right\}$$

$$J = \left(\begin{array}{c|c} -i\frac{q}{p+q}\mathrm{Id}_p & \\ \hline & i\frac{p}{p+q}\mathrm{Id}_q \end{array}\right) \in \mathfrak{z} \subset \mathfrak{h}$$

$$\mathfrak{m}^{\mathbb{C}} = \left\{ \left(\begin{array}{c|c} 0 & E \\ \hline D & 0 \end{array}\right) \mid D \in \mathrm{Hom}(\mathbb{C}^p,\mathbb{C}^q), E \in \mathrm{Hom}(\mathbb{C}^q,\mathbb{C}^p) \right\}$$

$$\mathfrak{m}^{+} = \left\{ \left(\begin{array}{c|c} 0 & 0 \\ \hline D & 0 \end{array}\right) \mid D \in \mathrm{Hom}(\mathbb{C}^q,\mathbb{C}^p) \right\} = \mathrm{Hom}(\mathbb{C}^q,\mathbb{C}^p)$$

$$\mathfrak{m}^{-} = \left\{ \left(\begin{array}{c|c} 0 & E \\ \hline 0 & 0 \end{array}\right) \mid E \in \mathrm{Hom}(\mathbb{C}^p,\mathbb{C}^q) \right\} = \mathrm{Hom}(\mathbb{C}^p,\mathbb{C}^q)$$

$$\begin{aligned} G &= \mathrm{SU}(p,q) \\ H &= S(\mathrm{U}(p) \times \mathrm{U}(q)) \\ o(e^{2\pi J}) &= \frac{p+q}{gcd(p,q)} \end{aligned}$$

$$\mathfrak{c} = \{diag(a_1, \cdots, a_{p+q}) \in \mathfrak{su}(p,q)\}$$

Let $x_j \in \mathfrak{c}^*$ be such that $x_j(diag(a_1, \ldots, a_{p+q})) = a_j$.

$$\Delta = \{x_i - x_j\}_{i \neq j}$$

$$\Delta_C = \{x_i - x_j\}_{1 \leq i,j \leq q} \cup \{x_{p+i} - x_{p+j}\}_{1 \leq i,j \leq q}$$

$$\Delta_Q^+ = \{x_{p+i} - x_j\}_{1 \leq i \leq q, 1 \leq j \leq p}$$

$$\Gamma = \{\gamma_j = x_{p+j} - x_{p+1-j}\}_{1 \leq j \leq r}$$

Δ_Q^+		mult.		Δ_C
$x_{p+i} - x_{p+1-j}$ $x_{p+j} - x_{p+1-i}$	$\frac{1}{2}(\gamma_i + \gamma_j)$	2	$\frac{1}{2}(\gamma_i - \gamma_j)$	$x_{p+1-j} - x_{p+1-i}$ $x_{p+i} - x_{p+j}$
$x_{2p+k} - x_{p+1-i}$ $1 \leq k \leq q-p$	$\frac{1}{2}\gamma_i$	$q-p$	$-\frac{1}{2}\gamma_i$	$x_{2p+k} - x_{p+i}$ $1 \leq k \leq q-p$
		$(q-p)^2$	0	$x_{2p+i} - x_{2p+j}$ $1 \leq i,j \leq q-p$

$$N = 2(p-1) + (q-p) + 2 = p + q$$

Remark A.1. Roots in Δ_C are positive in the first row, negative in the second one, and both positive and negative in the third one.

Remark A.2. An example of projection:

$$\begin{aligned} x_{p+i} - x_j &= \frac{1}{2}\left[(x_{p+i} - x_{p+1-i}) + (x_{p+j} - x_{p+1-j}) + (x_{p+i} + x_{p+1-i}) + (x_{p+j} + x_{p+1-j})\right] \\ &= \frac{1}{2}(\gamma_i + \gamma_j) + \text{terms orthogonal to } \Gamma \end{aligned}$$

$\mathfrak{sp}(2n,\mathbb{R})$

$$\mathfrak{sp}(2n,\mathbb{R}) = \left\{ X \in \mathfrak{su}(n,n) \mid X^* \left(\begin{array}{c|c} & \mathrm{Id}_n \\ \hline -\mathrm{Id}_n & \end{array}\right) + \left(\begin{array}{c|c} & \mathrm{Id}_n \\ \hline -\mathrm{Id}_n & \end{array}\right) X = 0 \right\} =$$
$$= \left\{ \left(\begin{array}{c|c} A & B \\ \hline \overline{B} & \overline{A} \end{array}\right) \mid A \in \mathfrak{u}(p), B \text{ symmetric} \right\}$$

$$\mathfrak{h} = \left\{ \left(\begin{array}{c|c} A & 0 \\ \hline 0 & \overline{A} \end{array}\right) \mid A \in \mathfrak{u}(n) \right\} = \mathfrak{u}(n)$$

$$\mathfrak{m} = \left\{ \left(\begin{array}{c|c} 0 & B \\ \hline B^* & 0 \end{array}\right) \mid B \text{ symmetric} \right\}$$

$$J = \left(\begin{array}{c|c} -\frac{i}{2}\,\mathrm{Id}_n & \\ \hline & \frac{i}{2}\,\mathrm{Id}_n \end{array}\right) \in \mathfrak{z} \subset \mathfrak{h}$$

$$\mathfrak{m}^{\mathbb{C}} = \left\{ \left(\begin{array}{c|c} 0 & E \\ \hline D & 0 \end{array}\right) \mid D, E \in \mathrm{Sym}(n,\mathbb{C}) \right\}$$

$$\mathfrak{m}^{+} = \left\{ \left(\begin{array}{c|c} 0 & 0 \\ \hline D & 0 \end{array}\right) \mid D \in \mathrm{Sym}(n,\mathbb{C}) \right\} = \mathrm{Sym}^2\,\mathbb{C}^{n\,*}$$

$$\mathfrak{m}^{-} = \left\{ \left(\begin{array}{c|c} 0 & E \\ \hline 0 & 0 \end{array}\right) \mid E \in \mathrm{Sym}(n,\mathbb{C}) \right\} = \mathrm{Sym}^2\,\mathbb{C}^{n}$$

$$G = \mathrm{Sp}(2n,\mathbb{R})$$
$$H = \mathrm{U}(n)$$
$$e^{2\pi J} = -\mathrm{Id}$$

$$\mathfrak{c} = \{diag(ia_1, \cdots, ia_n, -ia_1, \ldots, -ia_n) \in \mathfrak{so}^*(2n)\}$$

Let $x_j \in \mathfrak{c}^*$ be such that $x_j(diag(ia_1, \cdots, ia_n, -ia_1, \ldots, -ia_n)) = a_j$.

$$\Delta = \{\pm x_i \pm x_j\}_{1 \leq i \neq j \leq n} \cup \{\pm 2x_i\}_{1 \leq i \leq n}$$

$$\Delta_C = \{x_i - x_j\}_{1 \leq i,j \leq n}$$

$$\Delta_Q^+ = \{x_i + x_j\}_{1 \leq i \neq j \leq n}$$

$$\Gamma = \{\gamma_j = 2x_j\}_{1 \leq j \leq r}$$

Δ_Q^+		mult.		Δ_C
$x_i + x_j$	$\frac{1}{2}(\gamma_i + \gamma_j)$	1	$\frac{1}{2}(\gamma_i - \gamma_j)$	$x_i - x_j$
		0	0	

$$N = 1(n-1) + 0 + 2 = n + 1$$

Remark A.3. Roots in Δ_C are positive in the first row for $i < j$.

Remark A.4. An example of projection:

$$x_i + x_j = \frac{1}{2}\left[2x_i + 2x_j\right] = \frac{1}{2}(\gamma_i + \gamma_j)$$

$\mathfrak{so}^*(2n)$

$$\mathfrak{so}^*(2n) = \left\{ X \in \mathfrak{sl}(p+q,\mathbb{C}) \mid X^* \left(\begin{array}{c|c} & \mathrm{Id}_n \\ \hline \mathrm{Id}_n & \end{array} \right) + \left(\begin{array}{c|c} & \mathrm{Id}_n \\ \hline \mathrm{Id}_n & \end{array} \right) X = 0 \right\} =$$

$$= \left\{ \left(\begin{array}{c|c} A & B \\ \hline B^* & \overline{A} \end{array} \right) \mid A \in \mathfrak{u}(p), B \in \mathfrak{so}(n,\mathbb{C}) \right\}$$

$$\mathfrak{h} = \left\{ \left(\begin{array}{c|c} A & 0 \\ \hline 0 & \overline{A} \end{array} \right) \mid A \in \mathfrak{u}(n) \right\} = \mathfrak{u}(n)$$

$$\mathfrak{m} = \left\{ \left(\begin{array}{c|c} 0 & B \\ \hline B^* & 0 \end{array} \right) \mid B \in \mathfrak{so}(n,\mathbb{C}) \right\}$$

$$J = \left(\begin{array}{c|c} -\frac{i}{2}\,\mathrm{Id}_n & \\ \hline & \frac{i}{2}\,\mathrm{Id}_n \end{array} \right) \in \mathfrak{z} \subset \mathfrak{h}$$

$$\mathfrak{m}^{\mathbb{C}} = \left\{ \left(\begin{array}{c|c} 0 & E \\ \hline D & 0 \end{array} \right) \mid D \in \mathfrak{so}(n,\mathbb{C}), E \in \mathfrak{so}(n,\mathbb{C}) \right\}$$

$$\mathfrak{m}^{+} = \left\{ \left(\begin{array}{c|c} 0 & 0 \\ \hline D & 0 \end{array} \right) \mid D \in \mathfrak{so}(n,\mathbb{C}) \right\} = \wedge^2 \mathbb{C}^{n\,*}$$

$$\mathfrak{m}^{-} = \left\{ \left(\begin{array}{c|c} 0 & E \\ \hline 0 & 0 \end{array} \right) \mid E \in \mathfrak{so}(n,\mathbb{C}) \right\} = \wedge^2 \mathbb{C}^{n}$$

$$\begin{aligned} G &= \mathrm{SO}^*(2n) \\ H &= \mathrm{U}(n) \\ e^{2\pi J} &= -\mathrm{Id} \end{aligned}$$

$$\mathfrak{c} = \{diag(ia_1, \cdots, ia_n, -ia_1, \ldots, -ia_n) \in \mathfrak{so}^*(2n)\}$$

Let $x_j \in \mathfrak{c}^*$ be such that $x_j(diag(ia_1, \cdots, ia_n, -ia_1, \ldots, -ia_n)) = a_j$.

We use the notation, $m = [n/2]$, $j' = 2m + 1 - j$.

$$\Delta = \{\pm x_i \pm x_j\}_{1 \leq i \neq j \leq n}$$

$$\Delta_C = \{x_i - x_j\}_{1 \leq i \neq j \leq n}$$

$$\Delta_Q^+ = \{x_i + x_j\}_{1 \leq i \neq j \leq n}$$

$$\Gamma = \{\gamma_j = x_j + x_{2m+1-j} = x_j + x_{j'}\}_{1 \leq i \leq m}$$

Δ_Q^+		mult.		Δ_C
$x_i + x_j$ $x_{i'} + x_j$ $x_i + x_{j'}$ $x_{i'} + x_{j'}$	$\frac{1}{2}(\gamma_i + \gamma_j)$	4	$\frac{1}{2}(\gamma_i - \gamma_j)$	$x_i - x_{j'}$ $x_{i'} - x_{j'}$ $x_i - x_j$ $x_{i'} - x_j$
$x_i + x_{2m+1}$ $x_{i'} + x_{2m+1}$	$\frac{1}{2}\gamma_i$	2	$-\frac{1}{2}\gamma_i$	$-x_{i'} + x_{2m+1}$ $-x_i + x_{2m+1}$
		$2m$	0	$\pm(x_i - x_{i'})$ $1 \leq i \leq m$

tube, $n = 2m$	$N = (m-1) + 0 + 2 = 4m - 2 = 2(n-1)$
non-tube, $n = 2m+1$	$N = 4(m-1) + 2 + 2 = 4m = 2(n-1)$

Remark A.5. Roots in Δ_C are positive in the first row for $i < j$, negative in the second one, and both positive and negative in the third one.

Remark A.6. An example of projection:

$$\begin{aligned} x_i + x_j &= \frac{1}{2}\left[(x_i - x_{i'}) + (x_j - x_{j'}) + (x_i + x_{i'}) + (x_j + x_{j'})\right] \\ &= \frac{1}{2}(\gamma_i + \gamma_j) + \text{terms orthogonal to } \Gamma \end{aligned}$$

$\mathfrak{so}(2,n)$

$$\mathfrak{so}(2,n) = \left\{X \in \mathfrak{gl}(2+n,\mathbb{R}) \mid X^t\left(\begin{array}{c|c} \mathrm{Id}_n & \\ \hline & -\mathrm{Id}_n \end{array}\right) + \left(\begin{array}{c|c} \mathrm{Id}_n & \\ \hline & -\mathrm{Id}_n \end{array}\right)X = 0\right\}$$

$$= \left\{\left(\begin{array}{c|c} A & B \\ \hline B^t & D \end{array}\right) \mid A \in \mathfrak{so}(2,\mathbb{R}), D \in \mathfrak{so}(n,\mathbb{R}), B \text{ arbitrary } 2\times n \text{ matrix}\right\}$$

$$\mathfrak{h} = \left\{\left(\begin{array}{c|c} A & 0 \\ \hline 0 & D \end{array}\right) \mid A \in \mathfrak{so}(2,\mathbb{R}), D \in \mathfrak{so}(n,\mathbb{R}),\right\} = \mathfrak{so}(2,\mathbb{R}) \oplus \mathfrak{so}(n,\mathbb{R})$$

$$\mathfrak{m} = \left\{\left(\begin{array}{c|c} 0 & B \\ \hline B^t & 0 \end{array}\right) \mid B \in \mathrm{Mat}_{2\times n}(\mathbb{R})\right\}$$

$$J = \left(\begin{array}{c|c} \left(\begin{array}{c|c} & 1 \\ \hline -1 & \end{array}\right) & \\ \hline & 0 \end{array}\right) \in \mathfrak{z} \subset \mathfrak{h}$$

$$\mathfrak{m}^{\mathbb{C}} = \left\{\left(\begin{array}{c|c} 0 & E \\ \hline D & 0 \end{array}\right) \mid D \in \mathrm{Mat}_{n\times 2}(\mathbb{C}), E \in \mathrm{Mat}_{2\times n}(\mathbb{C})\right\}$$

$$\mathfrak{m}^{+} = \left\{\left(\begin{array}{c|c} 0 & 0 \\ \hline D & 0 \end{array}\right) \mid D \in \mathrm{Mat}_{n\times 2}(\mathbb{C})\right\}$$

$$\mathfrak{m}^{-} = \left\{\left(\begin{array}{c|c} 0 & E \\ \hline 0 & 0 \end{array}\right) \mid E \in \mathrm{Mat}_{2\times n}(\mathbb{C})\right\}$$

$$\begin{aligned} G &= \mathrm{SO}_0(2,n) \\ H &= \mathrm{SO}(2) \times SO(n) \\ e^{2\pi J} &= -\mathrm{Id} \end{aligned}$$

$\mathfrak{c} = \{diag(a_1, \cdots, a_{n+2}) \in \mathfrak{so}(2, n)\}$

Let $x_j \in \mathfrak{c}^*$ be such that $x_j(diag(a_1, \cdots, a_{n+2})) = a_j$.

We use the notation, $m = 1 + [n/2]$ and n even / (n odd).

$\Delta = \{\pm x_i \pm x_j\}_{1 \leq i \neq j \leq m} \quad (\cup\{\pm x_i\}_{1 \leq i \leq m})$

$\Delta_C = \{\pm x_i \pm x_j\}_{2 \leq i \neq j \leq m} \, (\cup\{\pm x_i\}_{2 \leq i \leq m})$

$\Delta_Q^+ = \{x_1 \pm x_j\}_{2 \leq j \leq n} \quad (\cup\{x_1\})$

$\Gamma = \{\gamma_1 = x_1 + x_2, \gamma_2 = x_1 - x_2\}$

Δ_Q^+		mult.		Δ_C
$x_1 \pm x_j$ $1 \leq j \leq m$ (x_1)	$\frac{1}{2}(\gamma_1 + \gamma_2)$	$n - 2$	$\frac{1}{2}(\gamma_1 - \gamma_2)$	$x_2 \pm x_j$ $1 \leq j \leq m$ (x_2)
		$n(m-2)$	0	$\pm x_i \pm x_j$ $i, j \neq 1, 2$ $(\pm x_i)$ $i \neq 1, 2$

$$N = (n - 2)(2 - 1) + 0 + 2 = n$$

Remark A.7. Roots in Δ_C are positive in the first row, and both positive and negative in the second one.

Remark A.8. An example of projection:

$$\begin{aligned} x_1 \pm x_j &= \frac{1}{2}\left[(x_1 + x_2) + (x_1 - x_2))\right] \pm x_j \\ &= \frac{1}{2}(\gamma_1 + \gamma_2) + \text{terms orthogonal to } \Gamma \end{aligned}$$

$\mathfrak{e}^{6}_{-14}$

$$\Delta = \{\pm e_i \pm e_j\}_{1\le i\ne j\le 5} \cup \{\pm(e_8 - e_7 - e_6) + {}^1\!/\!_2 \sum_{j=1}^{5} (-1)^{\epsilon_j} e_i \mid \sum \epsilon_j \text{ is even } \}$$

$$\Delta_C = \{\pm e_i \pm e_j\}_{1\le i\ne j\le 5}$$

$$\Delta_Q^+ = \{e_8 - e_7 - e_6 + {}^1\!/\!_2 \sum_{j=1}^{8} (-1)^{\epsilon_j} e_i \mid \sum \epsilon_j \text{ is even } \}$$

$$R = \{{}^1\!/\!_2(e_8 - e_7 - e_6 - e_5 - e_4 - e_3 - e_2 + e_1), e_2 + e_1, e_2 - e_1, e_3 - e_2, e_4 - e_3, e_5 - e_4\}$$

Following [Kna02], we take the only non-compact root of R,

$$\gamma_1 = {}^1\!/\!_2(e_8 - e_7 - e_6 - e_5 - e_4 - e_3 - e_2 + e_1),$$

and look for the lowest orthogonal root

$$\gamma_2 = {}^1\!/\!_2(e_8 - e_7 - e_6 - e_5 - e_4 - e_3 + e_2 - e_1).$$

$$\Gamma = \{\gamma_1, \gamma_2\}$$

Δ_Q^+		m.		Δ_C
${}^1\!/\!_2(e_8 - e_7 - e_6 + e_5 - e_4 + e_3 + e_2 - e_1)$	$\frac{1}{2}(\gamma_1+\gamma_2)$	6	$\frac{1}{2}(\gamma_1-\gamma_2)$	$e_5 + e_3$
$- e_3 - e_2$				$e_5 - e_2$
$- e_5 + e_4 + e_3 + e_2$				$e_4 + e_3$
$- e_3 - e_2$				$e_4 - e_2$
$+ e_5 + e_4 - e_3 + e_2$				$e_5 + e_4$
$- e_5 - e_4 + e_3 - e_2$				$e_3 - e_2$
${}^1\!/\!_2(e_8 - e_7 - e_6 - e_5 + e_4 + e_3 - e_2 + e_1)$	$\frac{1}{2}\gamma_1$	4	$-\frac{1}{2}\gamma_1$	$-e_5 + e_1$
$+ e_5 - e_4$				$-e_4 + e_1$
$+ e_5 + e_4 - e_3 - e_2$				$-e_3 + e_1$
$+ e_3 - e_2$				$e_2 + e_1$
${}^1\!/\!_2(e_8 - e_7 - e_6 + e_5 - e_4 - e_3 + e_2 + e_1)$	$\frac{1}{2}\gamma_2$	4	$-\frac{1}{2}\gamma_2$	$e_5 + e_1$
$- e_5 + e_4$				$e_4 + e_1$
$- e_5 - e_4 + e_3 + e_2$				$e_3 + e_1$
$- e_3 - e_2$				$-e_2 + e_1$
		12	0	$\pm(e_5 - e_4)$
				$\pm(e_5 - e_3)$
				$\pm(e_5 + e_2)$
				$\pm(e_4 - e_3)$
				$\pm(e_4 + e_2)$
				$\pm(e_3 + e_2)$

$$N = 6(2-1) + 4 + 2 = 12$$

$\mathfrak{e}^{7}_{-25}$

$$\Delta = \{\pm(e_8 - e_7)\} \cup \{\pm e_i \pm e_j\}_{1\le i\neq j\le 6} \cup \{\pm(e_8 - e_7) + {}^1\!/_2 \sum_{j=1}^{6} (-1)^{\epsilon_j} e_i \mid \sum \epsilon_j \text{ is odd } \}$$

$$\Delta_C = \{\pm e_i \pm e_j\}_{1\le i\neq j\le 5} \cup \{\pm(e_8 - e_7 - e_6) + {}^1\!/_2 \sum_{j=1}^{5} (-1)^{\epsilon_j} e_i \mid \sum \epsilon_j \text{ is even } \}$$

$$\Delta_Q^+ = \{e_8 - e_7\} \cup \{e_6 \pm e_i\}_{1\le i\le 5} \cup \{(e_8 - e_7 + e_6) + {}^1\!/_2 \sum_{j=1}^{5} (-1)^{\epsilon_j} e_i \mid \sum \epsilon_j \text{ is odd } \}\}$$

Following [Kna02], we take the lowest root in Δ_Q^+,

$$\gamma_1 = {}^1\!/_2(e_8 - e_7 + e_6 + e_5 + e_4 + e_3 + e_2 - e_1),$$

and look for the lowest orthogonal root to γ_1,

$$\gamma_2 = e_6 - e_2$$

and again for the lowest root orthogonal to γ_1 and γ_2,

$$\gamma_3 = {}^1\!/_2(e_8 - e_7 + e_6 - e_5 - e_4 - e_3 + e_2 + e_1).$$

$$\Gamma = \{\gamma_1, \gamma_2, \gamma_3\}$$

$$N = 8(3-1) + 0 + 2 = 18$$

Δ_Q^+		m.		Δ_C
$1/2(e_8 - e_7 + e_6 + e_5 + e_4 + e_3 - e_2 + e_1)$ $- e_3 - e_2 - e_1$ $- e_4 + e_3 - e_2 - e_1$ $- e_5 + e_4 + e_3 - e_2 - e_1$ $e_6 + e_3,\ e_6 + e_4,\ e_6 + e_5,\ e_6 - e_1$	$\frac{1}{2}(\gamma_1 + \gamma_2)$	8	$\frac{1}{2}(\gamma_1 - \gamma_2)$	$1/2(e_8 - e_7 - e_6 + e_5 + e_4 + e_3 + e_2 + e_1)$ $- e_3 + e_2 - e_1$ $- e_4 + e_3 + e_2 - e_1$ $- e_5 + e_4 + e_3 + e_2 - e_1$ $e_2 + e_3,\ e_2 + e_4,\ e_2 + e_5,\ e_2 - e_1$
$1/2(e_8 - e_7 + e_6 - e_5 - e_4 - e_3 - e_2 - e_1)$ $+ e_3 - e_2 + e_1$ $+ e_4 - e_3 - e_2 + e_1$ $+ e_5 - e_4 - e_3 - e_2 + e_1$ $e_6 - e_3$ $e_6 - e_4$ $e_6 - e_5$ $e_6 + e_1$	$\frac{1}{2}(\gamma_2 + \gamma_3)$	8	$\frac{1}{2}(\gamma_2 - \gamma_3)$	$-e_2 - e_1$ $-e_3 - e_1$ $-e_4 - e_1$ $-e_5 - e_1$ $-1/2(e_8 - e_7 - e_6 - e_5 - e_4 + e_3 + e_2 + e_1)$ $+ e_4 - e_3$ $+ e_5 - e_4 + e_3$ $-1/2(e_8 - e_7 - e_6 - e_5 - e_4 - e_3 + e_2 - e_1$
$1/2(e_8 - e_7 + e_6 + e_5 - e_4 + e_3 + e_2 + e_1)$ $- e_5 + e_4$ $e_5 + e_4 - e_3 + e_2 + e_1$ $- e_5 + e_4 - e_3 - e_2 - e_1$ $- e_5 - e_4 + e_3 - e_2 - e_1$ $e_5 - e_4 - e_3 - e_2 - e_1$ $e_6 + e_2$ $e_8 - e_7$	$\frac{1}{2}(\gamma_1 + \gamma_3)$	8	$\frac{1}{2}(\gamma_2 - \gamma_3)$	$e_5 + e_3$ $e_4 + e_3$ $e_5 + e_4$ $e_4 - e_1$ $e_3 - e_1$ $e_5 - e_1$ $-1/2(e_8 - e_7 - e_6 - e_5 - e_4 - e_3 - e_2 + e_1)$ $1/2(e_8 - e_7 - e_6 + e_5 + e_4 + e_3 - e_2 - e_1)$
		24	0	$\pm(e_5 - e_4),\ \pm(e_5 - e_3),\ \pm(e_5 - e_2)$ $\pm(e_4 - e_3),\ \pm(e_4 - e_2),\ \pm(e_3 - e_2)$ $\pm 1/2(e_8 - e_7 - e_6 + e_5 \pm (e_4 - e_3) - e_2 + e_1)$ $\pm 1/2(e_8 - e_7 - e_6 - e_5 \pm (e_4 - e_3) - e_2 - e_1)$ $\pm 1/2(e_8 - e_7 - e_6 - e_5 + e_4 + e_3 - e_2 + e_1)$ $\pm 1/2(e_8 - e_7 - e_6 + e_5 - e_4 - e_3 - e_2 - e_1)$

Appendix B

Additional remarks

Remark B.1. We list two alternative ways to define the dual Coxeter number N in a more general context. On one hand, in an irreducible root system R (or the corresponding simple real or complex Lie groups or algebras), the Coxeter number is defined as the number of roots divided by the rank, or equivalently, $M = \dim R / \operatorname{rk} R - 1$. If the highest root is written as $\theta = \sum a_i \alpha_i$ for a system of simple roots α_i, then the Coxeter number is $M = 1 + \sum a_i$. For the **dual Coxeter number** N we consider the dual of the highest root (which is not always the highest coroot, but the highest short coroot), $\theta^\vee = \sum a_i^\vee \alpha_i^\vee$ in the dual base of simple roots and we define it as $N = 1 + \sum a_i^\vee$. The Coxeter number and the dual Coxeter number coincide if the Dynkin diagram is simply laced (type A-D-E). On the other hand, let Δ be an irreducible reduced root system of rank R. There are at most two different lengths of roots. If there are two, we distinguish between long and short roots. Otherwise, all the roots are considered long roots. Let n_L and n_S be the number or long and short roots, respectively. Then,

$$N = \frac{2n_L + n_S}{2R} \text{ for } \Delta \neq G_2,$$

$$N = \frac{3n_L + n_S}{3R} \text{ for } \Delta = G_2.$$

This statement can be checked case by case. The difference in the case of G_2 is that the ratio between the length of long an short roots is $\sqrt{3}$ instead of $\sqrt{2}$. As an example, for the root system of type A_n, we have $((n+1)^2 - 1) - n = n^2 + n$ roots, and rank n, so the Coxeter number is $n - 1$. The dual Coxeter number is also $n - 1$.

Furthermore, the Coxeter number coincides with the dual Coxeter number except for the cases of $\mathrm{Sp}(2n, \mathbb{R})$ and $\mathrm{SO}_0(2, 2m+1)$, in which equals $2n$ and $n+1$ respectively. These two are the cases of type B or C, which are the only ones having roots of different length.

Remark B.2. We have the following formulas for the multiplicities of the restricted root system:

$$\dim \mathfrak{h} = R + c + a \cdot r \cdot (r-1) + 2 \cdot b \cdot r,$$

$$\dim \mathfrak{m} = a \cdot r \cdot (r-1) + 2 \cdot b \cdot r + 2 \cdot r.$$

Remark B.3. Another possible approaches are mentioned in [FKK$^+$00], pp. 226-228. The determinant coincides with the Koecher norm function on the cone $\Omega \subset \mathfrak{n}_T^+$, which is defined as $\Delta(x) = c' \left(\int_\Omega e^{-(x|y)} dy \right)^{-r/n_T}$, with c' such that $\Delta(e_\Gamma) = 1$. Alternatively, we can define the determinant in $\mathfrak{a}_T^+$ and then extend it. For $\sum_{j=1}^r \lambda_j e_{\gamma_j} \in \mathfrak{a}_T^+$, we define $D_| \left(\sum_{j=1}^r \lambda_j e_{\gamma_j} \right) = \prod_{j=1}^r \lambda_j$. This polynomial is clearly invariant under the permutation group, which is shown to be the Weyl group W of H_T^*/H_0'. By a theorem by Chevalley, the algebra of $\mathrm{Ad}(H_T^*)$-invariant polynomials on $\mathfrak{n}_T^+$ is isomorphic to the W-invariant polynomials on $\mathfrak{a}_T^+$. Hence, the W-invariant polynomial $D_|$ extends uniquely to a polynomial D on $\mathfrak{n}_T^+$.

Remark B.4. We give the description of the determinant in the irreducible cases ([KV79], pg. 183, Remark 2). In the cases of $\mathrm{SU}(n,n)$ and $Sp(2n,\mathbb{R})$ the determinant function is the usual determinant, by looking at $\mathfrak{m}_+$ ($\mathrm{Hom}(\mathbb{C}^n, \mathbb{C}^n)$ and $S^2(\mathbb{C}^n)$ respectively) as endomorphims of a vector space. In both cases, the degree of the determinant coincides with the rank of the symmetric space. In the case of $SO^*(4m)$, $\mathfrak{m}^+ = \Lambda^2(\mathbb{C}^{2m})$, and the determinant is the Pfaffian of the element seen as a matrix. As it is a skew-Hermitian $2m \times 2m$ matrix, the Pfaffian is the square root of the usual determinant, and it is a polynomail of degree m (the rank of the symmetric space). For the group $\mathrm{SO}_0(2,n)$, $\mathfrak{m}^+ = \mathrm{Hom}(\mathbb{C}^n, \mathbb{C})$ and the determinant is the squared norm, which has degree equal to the rank 2. In the case of E_7^{-25}, $\mathfrak{m}_+$ is the space of 3×3 Hermitian matrices over the bioctonions. A determinant in the usual way is defined, taking into account the non-commutativity and the non-associativity of the bioctonions. This determinant is a polynomial of degree 3, which is the rank of the exceptional symmetric space defined by E_7^{-25}.

Remark B.5. The importance of the centreless hypothesis... This can be easily seen if we write the elements of $SU(p,q)$ as matrices with respect to the standard representation. However, if we want to generalize this for all the groups, we should use a representation which we know that exist for all of them, namely, the adjoint representation. It happens that there is a patological case in which the adjoint representation does not distinguish between elements of C_G' and elements of $H_T^{\mathbb{C}}$, the group $SU(p,3p)$. Let λ be a $2p$-root of the unit. The diagonal matrices $(\lambda; \lambda, 1, 1)$

and $(1;1,\lambda^{-1},\lambda^{-1})$ are elements of $Z(H_T^{\mathbb{C}})$ and $Z(C'_G)$ respectively, but their images under the adjoint representation are the same.

Lemma B.6. *The representations* $\mathrm{Ad} : H^{\mathbb{C}} \to \mathrm{Aut}(\mathfrak{m}^{\pm})$ *have the same kernel and this kernel is contained in* $Z(G^{\mathbb{C}}) \cap Z(H^{\mathbb{C}})$.

Proof. The $\mathrm{Ad}(H^{\mathbb{C}})$-equivariant isomorphism φ_-^+ gives the equality of the kernels. Since $[\mathfrak{m}^+,\mathfrak{m}^-]=\mathfrak{h}^{\mathbb{C}}$, we also have that the kernel acts trivially in $\mathfrak{h}^{\mathbb{C}}$, and therefore it is contained in the kernel of $\mathrm{Ad} : G^{\mathbb{C}} \to \mathrm{Aut}(\mathfrak{g}^{\mathbb{C}})$, which is $Z(G^{\mathbb{C}})$. □

Remark B.7. The rank of a Jordan algebra is defined as the maximal degree of the minimal polynomials of its elements. In some sources, as Roos in [FKK+00], p.476, the rank of an element is defined as the degree of its minimal polynomial. This definition does not coincide with the one just given. For instance, the unit e has maximal rank but minimal polynomial $X-e$. However, in other references, as [FK94], p.72, the rank in a simple Euclidean Jordan algebra is defined as the number of non-zero eigenvalues in its spectral decomposition. This definition does agree with ours.

Remark B.8. In general, the category of non-compact symmetric spaces is equivalent to the category of the so-called Hermitian positive Jordan triple systems. A Hermitian Jordan triple system is a complex vector space V endowed with a triple product $\{x,y,z\}\in V$, $\mathbb{C}$-bilinear in (x,z), $\mathbb{C}$-antilinear in y, symmetric in (x,z), i.e., $\{x,y,z\}=\{z,y,x\}$, and satisfying the Jordan identity

$$\{xy\{uvz\}\}-\{uv\{xyz\}\}=\{\{xyu\}vz\}-\{u\{vxy\}z\}.$$

If we fix an element $y\in V$ and consider the double product $(x,z)\mapsto\{x,y,z\}$ we obtain a Jordan algebra which is denoted by $V^{(y)}$. This algebra is not necessarily unital, but if we take any tripotent element ($\{e,e,e\}=e$), the set $V_2^{(e)}=\{z\in V \mid \{e,e,z\}=2z\}$ becomes a Jordan algebra with unit e. A Hermitian Jordan triple system is called positive if the form defined by $(x,y)\mapsto \mathrm{tr}(z\mapsto\{x,y,z\})$ is a Hermitian inner product. As a consequence, given a tripotent e, the real form $V_2^{(e),+}=\{z\in V^{(e)}\mid \{e,x,e\}=x\}$ of $V_2^{(e)}$ is a Euclidean unital Jordan algebra. Just as the Toledo character for groups of tube-type describes the semi-invariance of the determinant of the Jordan algebra, it would be very interesting to check the relation of the Toledo character for groups of non-tube type and the analogue of determinant for a Jordan triple system. We also mention that the dual Coxeter number as defined in Section 2.3.1 corresponds to the so-called genus of the Jordan triple.

Appendix C

Tables

G	H	$H^{\mathbb{C}}$	$\mathfrak{m}^{\mathbb{C}} = \mathfrak{m}^+ + \mathfrak{m}^-$
$\mathrm{SU}(p,q)$	$\mathrm{S}(\mathrm{U}(p) \times \mathrm{U}(q))$	$\mathrm{S}(\mathrm{GL}(p,\mathbb{C}) \times \mathrm{GL}(q,\mathbb{C}))$	$\mathrm{Hom}(\mathbb{C}^q,\mathbb{C}^p) + \mathrm{Hom}(\mathbb{C}^p,\mathbb{C}^q)$
$\mathrm{Sp}(2n,\mathbb{R})$	$\mathrm{U}(n)$	$\mathrm{GL}(n,\mathbb{C})$	$S^2(\mathbb{C}^n) + S^2(\mathbb{C}^{n*})$
$\mathrm{SO}^*(2n)$	$\mathrm{U}(n)$	$\mathrm{GL}(n,\mathbb{C})$	$\Lambda^2(\mathbb{C}^n) + \Lambda^2(\mathbb{C}^{n*})$
$\mathrm{SO}_0(2,n)$	$\mathrm{SO}(2) \times \mathrm{SO}(n)$	$\mathrm{SO}(2,\mathbb{C}) \times \mathrm{SO}(n,\mathbb{C})$	$\mathrm{Hom}(\mathbb{C}^n,\mathbb{C}) + \mathrm{Hom}(\mathbb{C},\mathbb{C}^n)$
E_6^{-14}	$\mathrm{Spin}(10) \times_{\mathbb{Z}_4} \mathrm{U}(1)$	$\mathrm{Spin}(10,\mathbb{C}) \times_{\mathbb{Z}_4} \mathbb{C}^*$	$\Delta_{10}^+ \otimes \eta^3 + \Delta_{10}^- \otimes \eta^{-3}$
E_7^{-25}	$\mathrm{E}_6^{-78} \times_{\mathbb{Z}_3} \mathrm{U}(1)$	$\mathrm{E}_6 \times_{\mathbb{Z}_3} \mathbb{C}^*$	$M \otimes \eta^2 + M^* \otimes \eta^{-2}$

Table C.1: Irreducible Hermitian symmetric spaces G/H

Remark C.1. We use the following notation:

- $\Delta_{10}^{\pm}$ are the half-spinor representations of the group $\mathrm{Spin}(10,\mathbb{C})$. They are 16-dimensional.
- M and M^* are the irreducible 27-dimensional representations of E_6, which are dual to each other.
- η^r is the representation $\eta^r : \mathbb{C}^* \to \mathrm{GL}(\mathbb{C}) \cong \mathbb{C}^*$ given by $z \mapsto z^r$.

G	$\mathrm{SU}(p,q)$	$\mathrm{Sp}(2n,\mathbb{R})$	$\mathrm{SO}^*(2n)$	$\mathrm{SO}_0(2,n)$
(E,φ) $\varphi=\beta+\gamma$	V: rank p bundle W: rank q bundle $\det V\otimes\det W=\mathcal{O}$ $\beta\in H^0(\mathrm{Hom}(W,V)\otimes K)$ $\gamma\in H^0(\mathrm{Hom}(V,W)\otimes K)$	V: rank n bundle $\beta\in H^0(S^2V\otimes K)$ $\gamma\in H^0(S^2V^*\otimes K)$	V: rank n bundle $\beta\in H^0(\Lambda^2V\otimes K)$ $\gamma\in H^0(\Lambda^2V^*\otimes K)$	$\left(V=L\oplus L^{-1}, Q_V=\left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right)\right)$ (W,Q_W): rank n orthogonal bundle L: line bundle; $\det W=\mathcal{O}$ $\beta\in H^0(\mathrm{Hom}(W,L)\otimes K)$ $\gamma\in H^0(\mathrm{Hom}(W,L^{-1})\otimes K)$
$G^{\mathbb{C}}\subset\mathrm{SL}(N,\mathbb{C})$ $\mathrm{E}=E(\mathbb{C}^N)$ $\Phi\in H^0(\mathrm{End}\,\mathrm{E}\otimes K)$	$\mathrm{E}=V\oplus W$ $\Phi=\left(\begin{smallmatrix}0&\beta\\\gamma&0\end{smallmatrix}\right)$	$\mathrm{E}=V\oplus V^*$ $\Phi=\left(\begin{smallmatrix}0&\beta\\\gamma&0\end{smallmatrix}\right)$	$\mathrm{E}=V\oplus V^*$ $\Phi=\left(\begin{smallmatrix}0&\beta\\\gamma&0\end{smallmatrix}\right)$	$\mathrm{E}=V\oplus W$ $\Phi=\begin{pmatrix}0&0&\beta\\0&0&\gamma\\-\gamma^t&-\beta^t&0\end{pmatrix}$
Invariant d'	$d'=\deg V=-\deg W$	$d'=\deg V$	$d'=\deg V$	$d'=\deg L$
Milnor–Wood type inequality $\lvert d'\rvert\le d'_{\max}$	$\lvert d'\rvert\le\min\{p,q\}(g-1)$	$\lvert d'\rvert\le n(g-1)$	$\lvert d'\rvert\le[\frac{n}{2}](2g-2)$	$\lvert d'\rvert\le 2g-2$

Table C.2: ([BGG03])Higgs bundles for irreducible classical Hermitian symmetric spaces G/H

G	E_6^{-14}	E_7^{-25}
(E,φ) $\varphi=\beta+\gamma$	(V,Q_V): orthogonal rank 10 bundle S_+: rank 16 associated Spin bundle L: line bundle $\beta \in H^0(E(\Delta_{10}^+ \otimes \eta^3)\otimes K)$ $\gamma \in H^0(E(\Delta_{10}^- \otimes \eta^{-3})\otimes K)$	(V,C_V): rank 27 bundle with E_6 structure N: line bundle $\beta \in H^0(E(M \otimes \eta^2)\otimes K)$ $\gamma \in H^0(E(M^* \otimes \eta^{-2})\otimes K)$
$G^{\mathbb{C}} \subset \mathrm{SL}(N,\mathbb{C})$ $\mathrm{E}=E(\mathbb{C}^N)$ $\Phi \in H^0(\operatorname{End}\mathrm{E}\otimes K)$	$\mathrm{E}=L+VL^{-1/2}+S_+L^{-1}$ $\Phi = \left(\begin{array}{c\|c} & \langle -,\gamma\rangle \\ & P(-,\beta) \\ \hline \beta \quad .\gamma & \end{array}\right)$	$\mathrm{E}=(VN^{-1/3}+N)+(V^*N^{1/3}+N^{-1})$ $\Phi = \left(\begin{array}{cc\|cc} & & \bar{C}(\gamma,-,-) & \beta \\ & & \langle \beta,-\rangle & 0 \\ \hline \beta & \gamma & & \\ \langle \gamma,-\rangle & 0 & & \end{array}\right)$
Toledo invariant	$d=\deg L$	$d=\deg N$
Milnor–Wood inequality $\|d\| \le d_{\max}$	$\|d\| \le 2(g-1)$	$\|d\| \le 3(g-1)$

Table C.3: Higgs bundles for irreducible exceptional Hermitian symmetric spaces G/H

G	H	H^*	H'	$\check{S} = H/H'$	$\mathfrak{m}'$	$\mathfrak{m}'^{\mathbb{C}}$
$\mathrm{SU}(n,n)$	$\mathrm{S}(\mathrm{U}(n)\times\mathrm{U}(n))$	$\{A \in \mathrm{GL}(n,\mathbb{C}) \mid \det(A)^2 \in \mathbb{R}^+\}$	$\{A \in \mathrm{U}(n) \mid \det(A)^2 = 1\}$	$\mathrm{U}(n)$	$\mathrm{Herm}(n,\mathbb{C})$	$\mathrm{Mat}(n,\mathbb{C})$
$\mathrm{Sp}(2n,\mathbb{R})$	$\mathrm{U}(n)$	$\mathrm{GL}(n,\mathbb{R})$	$\mathrm{O}(n)$	$\mathrm{U}(n)/\mathrm{O}(n)$	$\mathrm{Sym}(n,\mathbb{R})$	$\mathrm{Sym}(n,\mathbb{C})$
$\mathrm{SO}^*(2n)\ , n=2m$	$\mathrm{U}(n)$	$\mathrm{U}^*(n)$	$\mathrm{Sp}(n)$	$\mathrm{U}(n)/\mathrm{Sp}(n)$	$\mathrm{Herm}(m,\mathrm{H})$	$\mathrm{Skew}(n,\mathbb{C})$
$\mathrm{SO}_0(2,n)$	$\mathrm{SO}(2)\times\mathrm{SO}(n)$	$\mathrm{SO}_0(1,1)\times\mathrm{SO}(1,n-1)$	$\mathrm{O}(n-1)$	$\dfrac{\mathrm{U}(1)\times S^{n-1}}{\mathbb{Z}_2}$	$\mathbb{R}\times\mathbb{R}^{n-1}$	$\mathbb{C}\times\mathbb{C}^{n-1}$
E_7^{-25}	$\mathrm{E}_6^{-78}\times_{\mathbb{Z}_3}\mathrm{U}(1)$	$\mathrm{E}_6^{-26}\ltimes\mathbb{R}^*$	$\mathrm{F}_4\times\mathbb{Z}_2$	$\dfrac{\mathrm{E}_6^{-78}\cdot\mathrm{U}(1)}{\mathrm{F}_4}$	$\mathrm{Herm}(m,\mathbb{O})$	$\mathrm{Mat}(n,\mathbb{O})$

Table C.4: Irreducible tube type Hermitian symmetric spaces G/H

G	H	H'	$\widetilde{G}$	$\widetilde{H}$	$\widetilde{H}'$	$H'' = H'/\widetilde{H}'$
$\mathrm{SU}(p,q)\ , p < q$	$\mathrm{S}(\mathrm{U}(p) \times \mathrm{U}(q))$	$\{(A,B) \in \mathrm{U}(p) \times \mathrm{U}(q-p) :$ $\det(A)^2 \det(B) = 1\}$ $\cong \mathrm{S}(\mathrm{U}(p) \times \mathrm{U}(q-p)) \rtimes \mathbb{Z}_2$	$\mathrm{SU}(p,p)$	$\mathrm{S}(\mathrm{U}(p) \times \mathrm{U}(p))$	$\mathrm{SU}(p) \rtimes \mathbb{Z}_2$	$\mathrm{S}(\mathrm{U}(1) \times \mathrm{U}(q-p))$
$\mathrm{SO}^*(4m+2)$	$\mathrm{U}(2m+1)$	$\mathrm{Sp}(2m) \times \mathrm{U}(1)$	$\mathrm{SO}^*(4m)$	$\mathrm{U}(2m)$	$\mathrm{Sp}(2m)$	$\mathrm{U}(1)$
E_6^{-14}	$\mathrm{Spin}(10) \times_{\mathbb{Z}_4} \mathrm{U}(1)$	H'	$\mathrm{Spin}_0(2,8)$	$\mathrm{Spin}(2)$ $\times\mathrm{Spin}(8)$	$\mathrm{Spin}_0(1,1)$ $\times\mathrm{Spin}(1,7)$	$\mathrm{U}(1)$

Table C.5: Irreducible non-tube Hermitian symmetric spaces G/H

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